Part 4 : "New Advanced Study of Magic Squares and Cubes"

Chapter 4 : Commentary Articles No.2: "Various Arts and Tools for Studying Magic Squares"

Section 1: Notations and 'Euler Squares' : Kanji Setsuda

#0. Modern Arts for Studying Magic Objects

In this section let me explain a little about some notations expressed by the positional writing system of numbers and about the concept of "Euler Squares".

They are really the important topics of modern arts of studying magic squares and cubes. Without having any knowledge of them, you might not get any of the master keys to open the secret doors inviting you to the wonderful world of magic objects.

#1. 'Classical Notation' and 'Mathematical'

There are various forms of expressing our concept of magic squares and cubes. I want you to see some examples listed below.

The one named 'Classical Notation' is one of the most familiar forms of all. It uses a series of natural numbers beginning with '1' for the numerical elements of our object. Of course, they are expressed in the conventional number system, 'Decimal'.

The next form named 'Mathematical Notation' uses the serial integers beginning with '0'. It is made after the classical one, always subtracting 1 from the value of each element. Many modern researchers prefer to use this style for their basic forms, since it is suitable to deal with the new number systems beside the decimal one.

```
/Classic      /Math.--.--.--.--.          /N2i                      /D2i
 1 12  7 14    6 13| 0|11| 6|13| 0 11     0000 1011 0110 1101       0101 0011 0110 0101
 8 13  2 11    1 10| 7|12| 1|10| 7 12     0111 1100 0001 1010       0101 1100 1001 1010
10  3 16  5   15  4| 9| 2|15| 4| 9  2     1001 0010 1111 0100       1010 0011 0110 1010
15  6  9  4    8  3|14| 5| 8| 3|14  5     1110 0101 1000 0011       1010 1100 1001 0101
                   '--'--'--'--'                                     *8/  *4/  *2/  *1/
```

The diagram '/N2i' in the center shows the new style of notation expressing every number in the positional number system of base 2, so called 'Binary Number System'.

The last '/D2i' diagram is made after the /N2i taking each value-part according to its position and placing it in the separate layers denoting their own positions: 2^3/, 2^2/, 2^1/ and 2^0/. I am afraid of my simple explanations confusing you, but I recommend you to make them actually by yourself, know them very well, and get used to them.

#2. What is the Positional Writing System of Numbers?

I don't intend to give you any educational lessons here to teach about the positional number system. Let me explain only about what is related to our magic square world. If you want to know more about it, you should ask to your own math teacher.

New number systems limit their usable numerical letters to the count of base N.

For instance, Decimal number system uses only ten numerical letters: {0, 1, 2, 3, 4, 5, 6, 7, 8 and 9} to express all values. You can express ten by '10' with the two letters '1' and '0'. We all know '10' means 1x10+0=10[Dec] and '25' means 2x10+5=25[Dec].

Positional number system of the Base 9 uses only nine numerical letters: {0, 1, 2, 3, 4, 5, 6, 7 and 8} to express all values. You can express nine by '10'. We know '10[N9i]' means 1x9+0=9[Dec], '25[N9i]' means 2x9+5=23[Dec] and '67[N9i]' means 6x9+7=61[Dec].

Positional number system of the Base 7 uses only seven numerical letters: {0, 1, 2, 3, 4, 5 and 6} to express all values. You can express seven by '10'. We know '10[N7i]' means 1x7+0

=7[Dec], '25[N7i]' means 2x7+5=19[Dec] and '63[N7i]' means 6x7+3=45[Dec].

Positional number system of the Base 5 uses only five numerical letters: {0, 1, 2, 3, and 4} to express all values. You can express five by '10'. We know '10[N5i]' means 1x5+0=5[Dec], '43[N5i]' means 4x5+3=23[Dec] and '213[N5i]' means 2x25+1x5+3=58[Dec].

Positional number system of the Base 3 uses only three numerical letters: {0, 1 and 2} to express all values. You can express three by '10'. We know '10[N3i]' means 1x3+0=3[Dec]. '21[N3i]' means 2x3+1=7[Dec] and '2010[N3i]' means 2x27+0x9+1x3+0=57[Dec].

Binary number system (of the Base 2) uses only two numerical letters: {0 and 1} to express all values. You can express two by '10'. We know '10[N2i]' means 1x2+0=2[Dec]. '101[N2i]' means 1x4+0x2+1=5[Dec] and '11010[N2i]' means 1x16+1x8+0x4+1x2+0=26[Dec].

** Correspondence among Various Notations **

Cls	Mat	N10i	N9i	N8i	N7i	N6i	N5i	N4i	N3i	N2i/
1	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2	2	10
4	3	3	3	3	3	3	3	3	10	11
5	4	4	4	4	4	4	4	10	11	100
6	5	5	5	5	5	5	10	11	12	101
7	6	6	6	6	6	10	11	12	20	110
8	7	7	7	7	10	11	12	13	21	111
9	8	8	8	10	11	12	13	20	22	1000
10	9	9	10	11	12	13	14	21	100	1001
11	10	10	11	12	13	14	20	22	101	1010
12	11	11	12	13	14	15	21	23	102	1011
13	12	12	13	14	15	20	22	30	110	1100
14	13	13	14	15	16	21	23	31	111	1101
15	14	14	15	16	20	22	24	32	112	1110
16	15	15	16	17	21	23	30	33	120	1111
17	16	16	17	20	22	24	31	100	121	10000
18	17	17	18	21	23	25	32	101	122	10001
19	18	18	20	22	24	30	33	102	200	10010
20	19	19	21	23	25	31	34	103	201	10011
21	20	20	22	24	26	32	40	110	202	10100
22	21	21	23	25	30	33	41	111	210	10101
23	22	22	24	26	31	34	42	112	211	10110
24	23	23	25	27	32	35	43	113	212	10111
25	24	24	26	30	33	40	44	120	220	11000
26	25	25	27	31	34	41	100	121	221	11001
27	26	26	28	32	35	42	101	122	222	11010
28	27	27	30	33	36	43	102	123	1000	11011
29	28	28	31	34	40	44	103	130	1001	11100
30	29	29	32	35	41	45	104	131	1002	11101
31	30	30	33	36	42	50	110	132	1010	11110
32	31	31	34	37	43	51	111	133	1011	11111
33	32	32	35	40	44	52	112	200	1012	100000
34	33	33	36	41	45	53	113	201	1020	100001
35	34	34	37	42	46	54	114	202	1021	100010
36	35	35	38	43	50	55	120	203	1022	100011
37	36	36	40	44	51	100	121	210	1100	100100
38	37	37	41	45	52	101	122	211	1101	100101
39	38	38	42	46	53	102	123	212	1102	100110
40	39	39	43	47	54	103	124	213	1110	100111
41	40	40	44	50	55	104	130	220	1111	101000
42	41	41	45	51	56	105	131	221	1112	101001
43	42	42	46	52	60	110	132	222	1120	101010

44	43	43	47	53	61	111	133	223	1121	101011
45	44	44	48	54	62	112	134	230	1122	101100
46	45	45	50	55	63	113	140	231	1200	101101
47	46	46	51	56	64	114	141	232	1201	101110
48	47	47	52	57	65	115	142	233	1202	101111
49	48	48	53	60	66	120	143	300	1210	110000
50	49	49	54	61	100	121	144	301	1211	110001
51	50	50	55	62	101	122	200	302	1212	110010
52	51	51	56	63	102	123	201	303	1220	110011
53	52	52	57	64	103	124	202	310	1221	110100
54	53	53	58	65	104	125	203	311	1222	110101
55	54	54	60	66	105	130	204	312	2000	110110
56	55	55	61	67	106	131	210	313	2001	110111
57	56	56	62	70	110	132	211	320	2002	111000
58	57	57	63	71	111	133	212	321	2010	111001
59	58	58	64	72	112	134	213	322	2011	111010
60	59	59	65	73	113	135	214	323	2012	111011
61	60	60	66	74	114	140	220	330	2020	111100
62	61	61	67	75	115	141	221	331	2021	111101
63	62	62	68	76	116	142	222	332	2022	111110
64	63	63	70	77	120	143	223	333	2100	111111
Cls	Mat	N10i	N9i	N8i	N7i	N6i	N5i	N4i	N3i	N2i/

They say Legendary Euler already knew about this style of notations and used these positional number systems 250 years ago, but it is in the present time of computer age that we really knew we all had to master these arts when we were school boys. Since electrical circuits in any computer can only know the difference between 'On' and 'Off', any basic software has to deal with the binary number system at first.

#3. How do you convert it to the one in another Number System?

You have to use the serial divisions by the base N to convert your decimal number to the one in another number system. We call the way as the 'decomposition by the base N'. See the next examples below. If you repeat your divisions by the same base N one after another, you can finally make your answer with all the remainders.

```
2)  2   <Decomposition of 2 by the Base 2>
 ------
2)  1 ... 0 ---------.
 ------              |
    0 ... 1 --------.|
              2[Dec]  =10[N2i]

<Composition> [1]x2+[0]=2[Dec]
          or  [1]x2^1+[0]x2^0=2[Dec]
```

```
2)  5   <Decomposition of 5 by the Base 2>
 ------
2)  2 ... 1 ----------.
 ------               |
2)  1 ... 0 ---------.|
 ------              ||
    0 ... 1 --------.||
              5[Dec]  =101[N2i]

<Composition> ([1]x2+[0])x2+[1]=5[Dec]
          or  [1]x2^2+[0]x2^1+[1]x2^0=5[Dec]
```

```
2) 49   <Decomposition of 49 by the Base 2>
 ------
2) 24 ... 1 ----------------.
 ------                     |
2) 12 ... 0 --------------. |
 ------                   | |
2)  6 ... 0 ------------. | |
 ------                 | | |
2)  3 ... 0 ----------. | | |
 ------               | | | |
2)  1 ... 1 --------. | | | |
 ------   .---------. | | | | |
    0 ... 1/   49[Dec]   = 110001[N2i]
<Composition> (((([1]x2+[1])x2+[0])x2+[0])x2+[0])x2+[1]=49[Dec]
          or  [1]x2^5+[1]x2^4+[0]x2^3+[0]x2^2+[0]x2^1+[1]x2^0=49[Dec]

3) 76   <Decomposition of 76 by the Base 3>
 ------
3) 25 ... 1
 ------
3)  8 ... 1
 ------
3)  2 ... 2
 ------
    0 ... 2   76[Dec]   = 2211[N3i]
<Composition> (([2]x3+[2])x3+[1])x3+[1]=76[Dec]
          or  [2]x3^3+[2]x3^2+[1]x3^1+[1]x3^0=76[Dec]

3) 80   <Decomposition of 80 by the Base 3>
 ------
3) 26 ... 2
 ------
3)  8 ... 2
 ------
3)  2 ... 2
 ------
    0 ... 2   80[Dec]   = 2222[N3i]
<Composition> (([2]x3+[2])x3+[2])x3+[2]=80[Dec]
          or  [2]x3^3+[2]x3^2+[2]x3^1+[2]x3^0=80[Dec]
```

If you want to make your result back into the original decimal number, you may well multiply each position by the base N and add the next position to the product. Repeat the same job one after another until you get to the last position. We call this way as the 'composition', since it means the reverse conversion of 'decomposition'.

$V_{[Dec]}=(((((AxN+B)xN+C)xN+D)xN+E)xN+F;$

or $V_{[Dec]}=AxN^5 + BxN^4 + CXN^3 + DxN^2 + ExN^1 + FxN^0;$

If you want to have your original values in the classical notation, you always have to add 1 to each result in the mathematical notation.

#4. Basic Forms of Mathematical Notation and /N5i

Many modern researchers prefer to use such the next forms as below for our magic squares of order 5. The second diagram is called '/N5i' (Notation of the 5th-increment number system). All the 25 numbers are expressed in the positional number system of the base 5, and are placed in the correct locations so that you could see the same value patterns with the x/y co-ordinates of the locations themselves.

Yes. The number patterns are just the same to each other between the values and locations. It is the basis of making some transformations possible, such as the tricky 'Do-it-After-the-Model Transformation' exchanging their information of those two.

```
 /Mathematical    x\y 0  1  2  3  4   Locations in x/y Co-ordinates
 .--------------.   .--------------.   00[N5i] is located in (0, 0).
 | 0| 1| 2| 3| 4|  0|00|01|02|03|04|   01[N5i] is located in (0, 1).
 |--+--+--+--+--|   |--+--+--+--+--|   02[N5i] is located in (0, 2).
 | 5| 6| 7| 8| 9|  1|10|11|12|13|14|   03[N5i] is located in (0, 3).
 |--+--+--+--+--|   |--+--+--+--+--|   04[N5i] is located in (0, 4).
 |10|11|12|13|14|  2|20|21|22|23|24|   10[N5i] is located in (1, 0).
 |--+--+--+--+--|   |--+--+--+--+--|   11[N5i] is located in (1, 1).
 |15|16|17|18|19|  3|30|31|32|33|34|    . . . . .
 |--+--+--+--+--|   |--+--+--+--+--|   42[N5i] is located in (4, 2).
 |20|21|22|23|24|  4|40|41|42|43|44|   43[N5i] is located in (4, 3).
 '--------------'   '--------------'   44[N5i] is located in (4, 4).
```

#5. Basic Forms of Various Notations, /Nbasei and /Dbasei

I prefer to use all the diagrams listed below for our basic forms of magic squares of every order from 3 to 9.

** The Basic Squares with Decomposed Forms **

#1 Basic Forms of Order 3

```
  1  2  3     0  1  2     00 01 02    0 0 0   0 1 2
  4  5  6     3  4  5     10 11 12    1 1 1   0 1 2
  7  8  9     6  7  8     20 21 22    2 2 2   0 1 2
 /Classical /Mathematical /N3i        /D3i
```

#2 Basic Forms of Order 4

```
  1  2  3  4     0  1  2  3    0000 0001 0010 0011   0000 0000 0011 0101
  5  6  7  8     4  5  6  7    0100 0101 0110 0111   0000 1111 0011 0101
  9 10 11 12     8  9 10 11    1000 1001 1010 1011   1111 0000 0011 0101
 13 14 15 16    12 13 14 15    1100 1101 1110 1111   1111 1111 0011 0101
 /Classical     /Mathematical   /N2i                 /D2i
```

#3 Basic Forms of Order 5

```
  1  2  3  4  5    0  1  2  3  4    00 01 02 03 04    0 0 0 0 0   0 1 2 3 4
  6  7  8  9 10    5  6  7  8  9    10 11 12 13 14    1 1 1 1 1   0 1 2 3 4
 11 12 13 14 15   10 11 12 13 14    20 21 22 23 24    2 2 2 2 2   0 1 2 3 4
 16 17 18 19 20   15 16 17 18 19    30 31 32 33 34    3 3 3 3 3   0 1 2 3 4
 21 22 23 24 25   20 21 22 23 24    40 41 42 43 44    4 4 4 4 4   0 1 2 3 4
 /Classical        /Mathematical     /N5i              /D5i
```

#4 Basic Forms of Order 6

```
  1  2  3  4  5  6   000000 000001 000010 000011 000100 000101
  7  8  9 10 11 12   000110 000111 001000 001001 001010 001011
 13 14 15 16 17 18   001100 001101 001110 001111 010000 010001
 19 20 21 22 23 24   010010 010011 010100 010101 010110 010111
 25 26 27 28 29 30   011000 011001 011010 011011 011100 011101
 31 32 33 34 35 36   011110 011111 100000 100001 100010 100011
 /Classical          /N2i
  0  1  2  3  4  5   000000 000000 000000 000011 001100 010101
  6  7  8  9 10 11   000000 000000 001111 110000 110011 010101
 12 13 14 15 16 17   000000 000011 111100 111100 001100 010101
 18 19 20 21 22 23   000000 111111 000000 001111 110011 010101
 24 25 26 27 28 29   000000 111111 111111 000011 001100 010101
 30 31 32 33 34 35   001111 110000 110000 110000 110011 010101
 /Mathematical       /D2i
```

#5 Basic Forms of Order 7

```
 1  2  3  4  5  6  7    00 01 02 03 04 05 06
 8  9 10 11 12 13 14    10 11 12 13 14 15 16
15 16 17 18 19 20 21    20 21 22 23 24 25 26
22 23 24 25 26 27 28    30 31 32 33 34 35 36
29 30 31 32 33 34 35    40 41 42 43 44 45 46
36 37 38 39 40 41 42    50 51 52 53 54 55 56
43 44 45 46 47 48 49    60 61 62 63 64 65 66
/Classical              /N7i
 0  1  2  3  4  5  6    0 0 0 0 0 0 0   0 1 2 3 4 5 6
 7  8  9 10 11 12 13    1 1 1 1 1 1 1   0 1 2 3 4 5 6
14 15 16 17 18 19 20    2 2 2 2 2 2 2   0 1 2 3 4 5 6
21 22 23 24 25 26 27    3 3 3 3 3 3 3   0 1 2 3 4 5 6
28 29 30 31 32 33 34    4 4 4 4 4 4 4   0 1 2 3 4 5 6
35 36 37 38 39 40 41    5 5 5 5 5 5 5   0 1 2 3 4 5 6
42 43 44 45 46 47 48    6 6 6 6 6 6 6   0 1 2 3 4 5 6
/Mathematical           /D7i
```

#6 Basic Forms of Order 8

```
 1  2  3  4  5  6  7  8   000000 000001 000010 000011 000100 000101 000110 000111
 9 10 11 12 13 14 15 16   001000 001001 001010 001011 001100 001101 001110 001111
17 18 19 20 21 22 23 24   010000 010001 010010 010011 010100 010101 010110 010111
25 26 27 28 29 30 31 32   011000 011001 011010 011011 011100 011101 011110 011111
33 34 35 36 37 38 39 40   100000 100001 100010 100011 100100 100101 100110 100111
41 42 43 44 45 46 47 48   101000 101001 101010 101011 101100 101101 101110 101111
49 50 51 52 53 54 55 56   110000 110001 110010 110011 110100 110101 110110 110111
57 58 59 60 61 62 63 64   111000 111001 111010 111011 111100 111101 111110 111111
/Classical                /N2i
 0  1  2  3  4  5  6  7   00000000 00000000 00000000 00001111 00110011 01010101
 8  9 10 11 12 13 14 15   00000000 00000000 11111111 00001111 00110011 01010101
16 17 18 19 20 21 22 23   00000000 11111111 00000000 00001111 00110011 01010101
24 25 26 27 28 29 30 31   00000000 11111111 11111111 00001111 00110011 01010101
32 33 34 35 36 37 38 39   11111111 00000000 00000000 00001111 00110011 01010101
40 41 42 43 44 45 46 47   11111111 00000000 11111111 00001111 00110011 01010101
48 49 50 51 52 53 54 55   11111111 11111111 00000000 00001111 00110011 01010101
56 57 58 59 60 61 62 63   11111111 11111111 11111111 00001111 00110011 01010101
/Mathematical             /D2i
```

#7 Basic Forms of Order 9

```
 1  2  3  4  5  6  7  8  9   0000 0001 0002 0010 0011 0012 0020 0021 1022
10 11 12 13 14 15 16 17 18   0100 0101 0102 0110 0111 0112 0120 0121 0122
19 20 21 22 23 24 25 26 27   0200 0201 0202 0210 0211 0212 0220 0221 0222
28 29 30 31 32 33 34 35 36   1000 1001 1002 1010 1011 1012 1020 1021 1022
37 38 39 40 41 42 43 44 45   1100 1101 1102 1110 1111 1112 1120 1121 1122
46 47 48 49 50 51 52 53 54   1200 1201 1202 1210 1211 1212 1220 1221 1222
55 56 57 58 59 60 61 62 63   2000 2001 2002 2010 2011 2012 2020 2021 2022
64 65 66 67 68 69 70 71 72   2100 2101 2102 2110 2111 2112 2120 2121 2122
73 74 75 76 77 78 79 80 81   2200 2201 2202 2210 2211 2212 2220 2221 2222
/Classical                   /N3i
 0  1  2  3  4  5  6  7  8   000000000 000000000 000111222 012012012
 9 10 11 12 13 14 15 16 17   000000000 111111111 000111222 012012012
18 19 20 21 22 23 24 25 26   000000000 222222222 000111222 012012012
27 28 29 30 31 32 33 34 35   111111111 000000000 000111222 012012012
36 37 38 39 40 41 42 43 44   111111111 111111111 000111222 012012012
45 46 47 48 49 50 51 52 53   111111111 222222222 000111222 012012012
```

```
54 55 56 57 58 59 60 61 62  222222222 000000000 000111222 012012012
63 64 65 66 67 68 69 70 71  222222222 111111111 000111222 012012012
72 73 74 75 76 77 78 79 80  222222222 222222222 000111222 012012012
/Mathematical               /D3i
```

#6. Further Study with /Dbasei Patterns

Let's study further in various forms actually with some samples of solutions next.

I want you to get used to the 'decomposed diagram' of the base N such as shown below. '/D3i' means the **D**ecompositions by the **3**rd-**i**ncrement number system.

It is made after the 'N3i' form, taking each value from /N3i, decomposing it by the base 3 and placing each value-part in the separate layer of /D3i for higher position or lower. /H diagram only collects the value-parts of higher positions, and /L collects the ones of lower only. Each keeps the same location information of its original.

*** Magic Squares 3x3 ***

```
                              .------.
                             /   .-----\-.
  /Classic  /Math           /.-/:======\=\=====:-.
   2  7  6   1  6  5        01 20 12    0 2 1    1 0 2
   9  5  1   8  4  0        22 11 00    2 1 0    2 1 0
   4  3  8   3  2  7        10 02 21    1 0 2    0 2 1
                            /N3i        D3i/H    /L
```

This /D3i form is quite useful for you to see and study the structure of our magic object. For instance, any kind of line sum is easily calculated in this form as follows.

*** Check Line-Sums ***

```
1+6+5=(0+2+1)*3+(1+0+2)=12 ... rw1| 1+8+3=(0+2+1)*3+(1+2+0)=12 ... cl1
8+4+0=(2+1+0)*3+(2+1+0)=12 ... rw2| 6+4+2=(2+1+0)*3+(0+1+2)=12 ... cl2
3+2+7=(1+0+2)*3+(0+2+1)=12 ... rw3| 5+0+7=(1+0+2)*3+(2+0+1)=12 ... cl3
1+4+7=(0+1+2)*3+(1+1+1)=12 ... pd1| 5+4+3=(1+1+1)*3+(2+1+0)=12 ... pb1
```

When you get used to it, you will surely be able to see and calculate those sums quickly by heart, since you only have to catch the same 'number pattern' there.

Line-sums of those kinds must be always equal to the magic constant in our object. You can easily check and know whether they are equal to that or not.

*** Decomposition of Various Magic Squares ***

**** Pan-diagonal Magic Squares 5x5 Decomposed ****

```
 1/Classic          /Math.--.--.--.--.--.
   1 23 20 14  7    13  6| 0|22|19|13| 6| 0 22
  15  9  2 21 18    20 17|14| 8| 1|20|17|14  8
  22 16 13 10  4     9  3|21|15|12| 9| 3|21 15
   8  5 24 17 11    16 10| 7| 4|23|16|10| 7  4
  19 12  6  3 25     2 24|18|11| 5| 2|24|18 11
                         '--'--'--'--'--'

  /N5i              /D5i  *5/          *1/
  00 42 34 23 11    0 4 3 2 1     0 2 4 3 1
  24 13 01 40 32    2 1 0 4 3     4 3 1 0 2
  41 30 22 14 03    4 3 2 1 0     1 0 2 4 3
  12 04 43 31 20    1 0 4 3 2     2 4 3 1 0
  33 21 10 02 44    3 2 1 0 4     3 1 0 2 4
```

*** Check Line-Sums ***

```
0+22+19+13+6= (0+4+3+2+1)*5+(0+2+4+3+1)= 60 ... rw1
14+8+1+20+17= (2+1+0+4+3)*5+(4+3+1+0+2)= 60 ... rw2
21+15+12+9+3= (4+3+2+1+0)*5+(1+0+2+4+3)= 60 ... rw3
```

```
7+4+23+16+10= (1+0+4+3+2)*5+(2+4+3+1+0)= 60 ... rw4
18+11+5+2+24= (3+2+1+0+4)*5+(3+1+0+2+4)= 60 ... rw5
0+14+21+7+18= (0+2+4+1+3)*5+(0+4+1+2+3)= 60 ... cl1
22+8+15+4+11= (4+1+3+0+2)*5+(2+3+0+4+1)= 60 ... cl2
19+1+12+23+5= (3+0+2+4+1)*5+(4+1+2+3+0)= 60 ... cl3
13+20+9+16+2= (2+4+1+3+0)*5+(3+0+4+1+2)= 60 ... cl4
6+17+3+10+24= (1+3+0+2+4)*5+(1+2+3+0+4)= 60 ... cl5
0+8+12+16+24= (0+1+2+3+4)*5+(0+3+2+1+4)= 60 ... pd1
22+1+9+10+18= (4+0+1+2+3)*5+(2+1+4+0+3)= 60 ... pd2
19+20+3+7+11= (3+4+0+1+2)*5+(4+0+3+2+1)= 60 ... pd3
13+17+21+4+5= (2+3+4+0+1)*5+(3+2+1+4+0)= 60 ... pd4
6+14+15+23+2= (1+2+3+4+0)*5+(1+4+0+3+2)= 60 ... pd5
0+17+9+23+11= (0+3+1+4+2)*5+(0+2+4+3+1)= 60 ... pb1
22+14+3+16+5= (4+2+0+3+1)*5+(2+4+3+1+0)= 60 ... pb2
19+8+21+10+2= (3+1+4+2+0)*5+(4+3+1+0+2)= 60 ... pb3
13+1+15+7+24= (2+0+3+1+4)*5+(3+1+0+2+4)= 60 ... pb4
6+20+12+4+18= (1+4+2+0+3)*5+(1+0+2+4+3)= 60 ... pb5
```

... This is certainly a Pan-diagonal Magic Square!

```
0+24=24  ... cp1; 22+2=24  ... cp2;  19+5=24 ... cp3;  13+11=24 ... cp4;
6+18=24  ... cp5; 14+10=24 ... cp6;  8+16=24 ... cp7;  1+23=24  ... cp8;
20+4=24  ... cp9; 17+7=24  ... cp10; 21+3=24 ... cp11; 15+9=24  ... cp12;
12+12=24 ... cp13;
```

... This is certainly a Self-complementary type!

** Pan-diagonal Magic Squares 7x7 Decomposed **

```
 1/Classic                 /Mathem .--.--.--.--.--.--.--.
  1 48 39 30 28 19 10      27 18  9|  0|47|38|29|27|18|  9|  0 47 38
 18  9  7 47 38 29 27      37 28 26|17|  8|  6|46|37|28|26|17  8  6
 35 26 17  8  6 46 37       5 45 36|34|25|16|  7|  5|45|36|34 25 16
 45 36 34 25 16 14  5      15 13  4|44|35|33|24|15|13|  4|44 35 33
 13  4 44 42 33 24 15      32 23 14|12|  3|43|41|32|23|14|12  3 43
 23 21 12  3 43 41 32      42 40 31|22|20|11|  2|42|40|31|22 20 11
 40 31 22 20 11  2 49      10  1 48|39|30|21|19|10|  1|48|39 30 21
                                   '--'--'--'--'--'--'--'

 /N7i                       /D7i        *7/              *1/
 00 65 53 41 36 24 12       0 6 5 4 3 2 1    0 5 3 1 6 4 2
 23 11 06 64 52 40 35       2 1 0 6 5 4 3    3 1 6 4 2 0 5
 46 34 22 10 05 63 51       4 3 2 1 0 6 5    6 4 2 0 5 3 1
 62 50 45 33 21 16 04       6 5 4 3 2 1 0    2 0 5 3 1 6 4
 15 03 61 56 44 32 20       1 0 6 5 4 3 2    5 3 1 6 4 2 0
 31 26 14 02 60 55 43       3 2 1 0 6 5 4    1 6 4 2 0 5 3
 54 42 30 25 13 01 66       5 4 3 2 1 0 6    4 2 0 5 3 1 6
```

* Check Line-Sums *

```
0+47+38+29+27+18+9=  (0+6+5+4+3+2+1)*7+(0+5+3+1+6+4+2)=168 ... rw1
17+8+6+46+37+28+26=  (2+1+0+6+5+4+3)*7+(3+1+6+4+2+0+5)=168 ... rw2
34+25+16+7+5+45+36=  (4+3+2+1+0+6+5)*7+(6+4+2+0+5+3+1)=168 ... rw3
44+35+33+24+15+13+4= (6+5+4+3+2+1+0)*7+(2+0+5+3+1+6+4)=168 ... rw4
12+3+43+41+32+23+14= (1+0+6+5+4+3+2)*7+(5+3+1+6+4+2+0)=168 ... rw5
22+20+11+2+42+40+31= (3+2+1+0+6+5+4)*7+(1+6+4+2+0+5+3)=168 ... rw6
39+30+21+19+10+1+48= (5+4+3+2+1+0+6)*7+(4+2+0+5+3+1+6)=168 ... rw7
0+17+34+44+12+22+39= (0+2+4+6+1+3+5)*7+(0+3+6+2+5+1+4)=168 ... cl1
47+8+25+35+3+20+30=  (6+1+3+5+0+2+4)*7+(5+1+4+0+3+6+2)=168 ... cl2
38+6+16+33+43+11+21= (5+0+2+4+6+1+3)*7+(3+6+2+5+1+4+0)=168 ... cl3
29+46+7+24+41+2+19=  (4+6+1+3+5+0+2)*7+(1+4+0+3+6+2+5)=168 ... cl4
27+37+5+15+32+42+10= (3+5+0+2+4+6+1)*7+(6+2+5+1+4+0+3)=168 ... cl5
18+28+45+13+23+40+1= (2+4+6+1+3+5+0)*7+(4+0+3+6+2+5+1)=168 ... cl6
9+26+36+4+14+31+48=  (1+3+5+0+2+4+6)*7+(2+5+1+4+0+3+6)=168 ... cl7
```

```
0+8+16+24+32+40+48=  (0+1+2+3+4+5+6)*7+(0+1+2+3+4+5+6)=168 ... pd1
47+6+7+15+23+31+39=  (6+0+1+2+3+4+5)*7+(5+6+0+1+2+3+4)=168 ... pd2
38+46+5+13+14+22+30= (5+6+0+1+2+3+4)*7+(3+4+5+6+0+1+2)=168 ... pd3
29+37+45+4+12+20+21= (4+5+6+0+1+2+3)*7+(1+2+3+4+5+6+0)=168 ... pd4
27+28+36+44+3+11+19= (3+4+5+6+0+1+2)*7+(6+0+1+2+3+4+5)=168 ... pd5
18+26+34+35+43+2+10= (2+3+4+5+6+0+1)*7+(4+5+6+0+1+2+3)=168 ... pd6
9+17+25+33+41+42+1=  (1+2+3+4+5+6+0)*7+(2+3+4+5+6+0+1)=168 ... pd7
0+26+45+15+41+11+30= (0+3+6+2+5+1+4)*7+(0+5+3+1+6+4+2)=168 ... pb1
47+17+36+13+32+2+21= (6+2+5+1+4+0+3)*7+(5+3+1+6+4+2+0)=168 ... pb2
38+8+34+4+23+42+19=  (5+1+4+0+3+6+2)*7+(3+1+6+4+2+0+5)=168 ... pb3
29+6+25+44+14+40+10= (4+0+3+6+2+5+1)*7+(1+6+4+2+0+5+3)=168 ... pb4
27+46+16+35+12+31+1= (3+6+2+5+1+4+0)*7+(6+4+2+0+5+3+1)=168 ... pb5
18+37+7+33+3+22+48=  (2+5+1+4+0+3+6)*7+(4+2+0+5+3+1+6)=168 ... pb6
9+28+5+24+43+20+39=  (1+4+0+3+6+2+5)*7+(2+0+5+3+1+6+4)=168 ... pb7
... This is certainly a Pan-diagonal Magic Square!
0+48=48  ... cp1;    47+1=48  ... cp2;    38+10=48 ... cp3;    29+19=48 ... cp4;
27+21=48 ... cp5;    18+30=48 ... cp6;    9+39=48  ... cp7;    17+31=48 ... cp8;
8+40=48  ... cp9;    6+42=48  ... cp10;   46+2=48  ... cp11;   37+11=48 ... cp12;
28+20=48 ... cp13;   26+22=48 ... cp14;   34+14=48 ... cp15;   25+23=48 ... cp16;
16+32=48 ... cp17;   7+41=48  ... cp18;   5+43=48  ... cp19;   45+3=48  ... cp20;
36+12=48 ... cp21;   44+4=48  ... cp22;   35+13=48 ... cp23;   33+15=48 ... cp24;
24+24=48 ... cp25;
... This is certainly a Self-complementary type!
```

Watch the two examples above and know that all line-sums are equal to the magic constant. True. Both of them are really the 'pandiagonal magic squares', as you see.

Could you notice that each line-sum has the same number pattern as follows?

V_{sum5}= (0+1+2+3+4)*5+(0+1+2+3+4)= 60 ... (E5) in PMS55

V_{sum7}= (0+1+2+3+4+5+6)*7+(0+1+2+3+4+5+6)= 168 ... (E7) in PMS77

Each line of each layer consists of {0, 1, 2, 3 and 4} in PMS55, and similarly {0, 1, 2, 3, 4, 5 and 6} in PMS77, and it uses any number strictly once in any line. It has neither repeating use nor drop-off of a certain number, as you see.

You don't have to care about the permutation of those elements, since it could never change its sum at all. You have to care for the only constant combination.

It is amazing that all line-sums could be expressed in the same form: 20 equations obey (E5), and all equations as many as 28 obey (E7) above. How beautiful they are!

But this is not my discovery at all. It has been known well in Europe for a very long time as the characteristic property of "Latin Squares" or "Greco-Latin Squares".

They say it is Legendary Leonhard Euler(1707-1783) that was so positive as to use this property for the purpose of composing any magic squares in modern forms.

#7. "Euler Squares" and "Complete Euler Squares"

It is a habit we used to call "Euler Square" for what has such properties as follows:

(1) Each line in each layer of decomposed form must consist of {0, 1, 2, ..., (N-1)}.
You must use any number strictly once in any line. You must not use any number twice or more often. You must not un-use it in any line, either.

(2) In the mathematical notation, any number of {0, 1, 2, ..., (N^2-1)} must be used once.
No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

V_{sumN}= {0+1+2+...+(N-1)}*N+{0+1+2+...+(N-1)}= Magic_Constant

You don't have to care about the permutation of {0, 1, 2, ..., (N-1)} here. You only have to care for the only constant combination of {0, 1, 2, ..., (N-1)}.

Certainly. Both PMS55 and PMS77 above are the examples of 'Euler Squares'.

I would like to call "Complete Euler Squares" for those beautiful magic squares, all of whose pandiagonals beside basic lines obey the 'Euler's Conditions' above.

The first example of MS33 is one of the 'Euler Squares', but it is not the 'Complete Euler Square', because its pandiagonals do not always obey the Euler's Conditions.

Legendary Leonhard Euler did not mention about pandiagonals and he might not have known even about 'Complete Euler Squares', Dr. Mutsumi Suzuki once told me.

In order 5, all 3600 pandiagonal magic squares are Complete Euler Squares, and all 16 Simultaneous magic squares: both self-complementary and pandiagonal are also Complete Euler Squares, I certified. It is wonderful all "Suzuki Squares" are found CES.

In order 7, I found 38102400 Complete Euler Squares among pandiagonal magic squares, and 3456 CES among Simultaneous magic squares: both self-complementary and pandiagonal. I was surprised at the big counts of those, and I was more shocked when I found there exist many 'Non-Euler Squares' among them, far more than CES.

#8. How about the Magic Squares of Even Orders?

Let's express some pandiagonal magic squares of order 4 and 8 in various forms next and know how many Complete Euler Squares we can find.

First of all let's make them in the decomposed forms by the base 4 and 8, and then by the base 2, that is, binary number system. What do they look like?

See the next examples below, the results of my recent calculations.

```
** Pan-diagonal Magic Squares 4x4 Decomposed by the Base 4 **
 1/Classic      /Math.--.--.--.--.        /N4i          /D4i
   1  8 13 12    12 11| 0| 7|12|11| 0  7    00 13 30 23    0 1 3 2  0 3 0 3
  14 11  2  7     1  6|13|10| 1| 6|13 10    31 22 01 12    3 2 0 1  1 2 1 2
   4  5 16  9    15  8| 3| 4|15| 8| 3  4    03 10 33 20    0 1 3 2  3 0 3 0
  15 10  3  6     2  5|14| 9| 2| 5|14  9    32 21 02 11    3 2 0 1  2 1 2 1
                     '--'--'--'--'                            *4/      *1/
* Check Line-Sums and See if they are equal to 30:
  0+7+12+11=(0+1+3+2)*4+(0+3+0+3)=30 ... rw1
  13+10+1+6=(3+2+0+1)*4+(1+2+1+2)=30 ... rw2
  3+4+15+8= (0+1+3+2)*4+(3+0+3+0)=30 ... rw3
  14+9+2+5= (3+2+0+1)*4+(2+1+2+1)=30 ... rw4
  0+13+3+14=(0+3+0+3)*4+(0+1+3+2)=30 ... cl1
  7+10+4+9= (1+2+1+2)*4+(3+2+0+1)=30 ... cl2
  12+1+15+2=(3+0+3+0)*4+(0+1+3+2)=30 ... cl3
  11+6+8+5= (2+1+2+1)*4+(3+2+0+1)=30 ... cl4
  0+10+15+5=(0+2+3+1)*4+(0+2+3+1)=30 ... pd1
  7+1+8+14= (1+0+2+3)*4+(3+1+0+2)=30 ... pd2
  12+6+3+9= (3+1+0+2)*4+(0+2+3+1)=30 ... pd3
  11+13+4+2=(2+3+1+0)*4+(3+1+0+2)=30 ... pd4
  0+6+15+9= (0+1+3+2)*4+(0+2+3+1)=30 ... pb1
  7+13+8+2= (1+3+2+0)*4+(3+1+0+2)=30 ... pb2
  12+10+3+5=(3+2+0+1)*4+(0+2+3+1)=30 ... pb3
  11+1+4+14=(2+0+1+3)*4+(3+1+0+2)=30 ... pb4
 ... This is certainly a Pan-diagonal Magic Square!
  0+5=5    ... cp1;   7+2=9    ... cp2;  12+9=21 ... cp3;   11+14=25 ... cp4;
  13+8=21  ... cp5;   10+15=25 ... cp6;  1+4=5   ... cp7;   6+3=9    ... cp8;

 2/Classic     /Math.--.--.--.--.       /N4i          /D4i
   1  8 11 14   10 13| 0| 7|10|13| 0  7   00 13 22 31    0 1 2 3  0 3 2 1
  12 13  2  7    1  6|11|12| 1| 6|11 12   23 30 01 12    2 3 0 1  3 0 1 2
   6  3 16  9   15  8| 5| 2|15| 8| 5  2   11 02 33 20    1 0 3 2  1 2 3 0
  15 10  5  4    4  3|14| 9| 4| 3|14  9   32 21 10 03    3 2 1 0  2 1 0 3
                    '--'--'--'--'
```

```
3/Classic      /Math.--.--.--.--.        /N4i          /D4i
  1 12  7 14    6 13| 0|11| 6|13| 0 11   00 23 12 31   0 2 1 3   0 3 2 1
  8 13  2 11    1 10| 7|12| 1|10| 7 12   13 30 01 22   1 3 0 2   3 0 1 2
 10  3 16  5   15  4| 9| 2|15| 4| 9  2   21 02 33 10   2 0 3 1   1 2 3 0
 15  6  9  4    8  3|14| 5| 8| 3|14  5   32 11 20 03   3 1 2 0   2 1 0 3
                    '--'--'--'--'
```

Although all of them are pandiagonal MS44, all the number patterns in all lines do not always obey the Euler's conditions. You can find such a pattern as {0, 0, 3, 3} or {1, 1, 2, 2} in some lines here and there. They are not our Complete Euler Squares at all.

I could find no CES among pandiagonal MS44 by the decompositions of base 4.

How about making pandiagonal MS44 by the decompositions of base 2?

I want you to see the next examples of the same pandiagonal solutions carefully.

**** Pan-diagonal Magic Squares 4x4 Decomposed by the Base 2 ****

```
1/Classic      /Math.--.--.--.--.        /N2i                    /D2i
  1  8 13 12   12 11| 0| 7|12|11| 0  7   0000 0111 1100 1011    0011 0110 0101 0101
 14 11  2  7    1  6|13|10| 1| 6|13 10   1101 1010 0001 0110    1100 1001 0101 1010
  4  5 16  9   15  8| 3| 4|15| 8| 3  4   0011 0100 1111 1000    0011 0110 1010 1010
 15 10  3  6    2  5|14| 9| 2| 5|14  9   1110 1001 0010 0101    1100 1001 1010 0101
                    '--'--'--'--'                                 *8/  *4/  *2/  *1/
* Check Line-Sums and See if they are equal to 30:
  0+7+12+11=(0+0+1+1)*8+(0+1+1+0)*4+(0+1+0+1)*2+(0+1+0+1)= 30 ... rw1
  13+10+1+6=(1+1+0+0)*8+(1+0+0+1)*4+(0+1+0+1)*2+(1+0+1+0)= 30 ... rw2
  3+4+15+8= (0+0+1+1)*8+(0+1+1+0)*4+(1+0+1+0)*2+(1+0+1+0)= 30 ... rw3
  14+9+2+5= (1+1+0+0)*8+(1+0+0+1)*4+(1+0+1+0)*2+(0+1+0+1)= 30 ... rw4
  0+13+3+14=(0+1+0+1)*8+(0+1+0+1)*4+(0+0+1+1)*2+(0+1+1+0)= 30 ... cl1
  7+10+4+9= (0+1+0+1)*8+(1+0+1+0)*4+(1+1+0+0)*2+(1+0+0+1)= 30 ... cl2
  12+1+15+2=(1+0+1+0)*8+(1+0+1+0)*4+(0+0+1+1)*2+(0+1+1+0)= 30 ... cl3
  11+6+8+5= (1+0+1+0)*8+(0+1+0+1)*4+(1+1+0+0)*2+(1+0+0+1)= 30 ... cl4
  0+10+15+5=(0+1+1+0)*8+(0+0+1+1)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pd1
  7+1+8+14= (0+0+1+1)*8+(1+0+0+1)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pd2
  12+6+3+9= (1+0+0+1)*8+(1+1+0+0)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pd3
  11+13+4+2=(1+1+0+0)*8+(0+1+1+0)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pd4
  0+6+15+9= (0+0+1+1)*8+(0+1+1+0)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pb1
  7+13+8+2= (0+1+1+0)*8+(1+1+0+0)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pb2
  12+10+3+5=(1+1+0+0)*8+(1+0+0+1)*4+(0+1+1+0)*2+(0+0+1+1)= 30 ... pb3
  11+1+4+14=(1+0+0+1)*8+(0+0+1+1)*4+(1+0+0+1)*2+(1+1+0+0)= 30 ... pb4
 ... This is certainly a Pan-diagonal Magic Square!
  0+5=5    ... cp1;   7+2=9     ... cp2;  12+9=21 ... cp3;   11+14=25 ... cp4;
  13+8=21 ... cp5;   10+15=25 ... cp6;  1+4=8   ... cp7;   6+3=9    ... cp8;
```

```
2/Classic      /Math.--.--.--.--.        /N2i                    /D2i
  1  8 11 14   10 13| 0| 7|10|13| 0  7   0000 0111 1010 1101    0011 0101 0110 0101
 12 13  2  7    1  6|11|12| 1| 6|11 12   1011 1100 0001 0110    1100 0101 1001 1010
  6  3 16  9   15  8| 5| 2|15| 8| 5  2   0101 0010 1111 1000    0011 1010 0110 1010
 15 10  5  4    4  3|14| 9| 4| 3|14  9   1110 1001 0100 0011    1100 1010 1001 0101
                    '--'--'--'--'                                 *8/  *4/  *2/  *1/
```

```
3/Classic      /Math.--.--.--.--.        /N2i                    /D2i
  1 12  7 14    6 13| 0|11| 6|13| 0 11   0000 1011 0110 1101    0101 0011 0110 0101
  8 13  2 11    1 10| 7|12| 1|10| 7 12   0111 1100 0001 1010    0101 1100 1001 1010
 10  3 16  5   15  4| 9| 2|15| 4| 9  2   1001 0010 1111 0100    1010 0011 0110 1010
 15  6  9  4    8  3|14| 5| 8| 3|14  5   1110 0101 1000 0011    1010 1100 1001 0101
                    '--'--'--'--'                                 *8/  *4/  *2/  *1/
```

Could you notice that all the lines including pandiagonals have the same number pattern {0, 0, 1, 1} in each layer of decompositions by binary number system?

What a beautiful structure they have! This characteristic property looks so beautiful that it should be noted for the new Euler's conditions, I believe. Let's modify a little about the previous Euler's Conditions as follows:

* New Euler's Conditions for Order 4 *

(1) Each line in each layer of /D2i forms must consist of {0, 0, 1, 1}. Number '0' must be used as often as '1' in any line. One must not be used more or less often than the other, nor must be un-used in any line.

(2) In the mathematical notation, any number of {0, 1, 2, ..., 14, 15} must be used once. No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

V_{sum4}= {0+0+1+1}*8+{0+0+1+1}*4+{0+0+1+1}*2+{0+0+1+1}=30 ... (E4)

You don't have to care about the permutation of {0, 0, 1, 1} here. It never changes the line sum. You only have to care for the only constant combination of {0, 0, 1, 1}.

If you accept these new conditions, you may now call all the examples above as 'Complete Euler Squares', since you can find the same number pattern even in their pandiagonals. The more CES of order 4, the better base of decompositions.

I could find 48 Complete Euler Squares of order 4 among all the pandiagonal magic squares 4x4. It means all the pandiagonal MS44 are Complete Euler Squares without any exceptions. There is no 'Non-Euler' type among pandiagonal solutions of order 4.

Let's examine the case of order 8 next, shall we?

** 'Composite & Complete' Magic Squares 8x8 with /D8i **

```
 1/Classic                   /Mathemat  .--.--.--.--.--.--.--.--.
  1 63  4 62  8 58  5 59      7 57  4 58| 0|62| 3|61| 7|57| 4|58| 0 62  3 61
 56 10 53 11 49 15 52 14     48 14 51 13|55| 9|52|10|48|14|51|13|55  9 52 10
 25 39 28 38 32 34 29 35     31 33 28 34|24|38|27|37|31|33|28|34|24 38 27 37
 48 18 45 19 41 23 44 22     40 22 43 21|47|17|44|18|40|22|43|21|47 17 44 18
 57  7 60  6 64  2 61  3     63  1 60  2|56| 6|59| 5|63| 1|60| 2|56  6 59  5
 16 50 13 51  9 55 12 54      8 54 11 53|15|49|12|50| 8|54|11|53|15 49 12 50
 33 31 36 30 40 26 37 27     39 25 36 26|32|30|35|29|39|25|36|26|32 30 35 29
 24 42 21 43 17 47 20 46     16 46 19 45|23|41|20|42|16|46|19|45|23 41 20 42
                                        '--'--'--'--'--'--'--'--'
 /N8i                        /D8i            *8/               *1/
 00 76 03 75 07 71 04 72     0 7 0 7 0 7 0 7  0 6 3 5 7 1 4 2
 67 11 64 12 60 16 63 15     6 1 6 1 6 1 6 1  7 1 4 2 0 6 3 5
 30 46 33 45 37 41 34 42     3 4 3 4 3 4 3 4  0 6 3 5 7 1 4 2
 57 21 54 22 50 26 53 25     5 2 5 2 5 2 5 2  7 1 4 2 0 6 3 5
 70 06 73 05 77 01 74 02     7 0 7 0 7 0 7 0  0 6 3 5 7 1 4 2
 17 61 14 62 10 66 13 65     1 6 1 6 1 6 1 6  7 1 4 2 0 6 3 5
 40 36 43 35 47 31 44 32     4 3 4 3 4 3 4 3  0 6 3 5 7 1 4 2
 27 51 24 52 20 56 23 55     2 5 2 5 2 5 2 5  7 1 4 2 0 6 3 5
```

*** Check Line-Sums ***

```
 0+62+3+61+7+57+4+58=      (0+7+0+7+0+7+0+7)*8+(0+6+3+5+7+1+4+2)= 252 ... rw1
 55+9+52+10+48+14+51+13=   (6+1+6+1+6+1+6+1)*8+(7+1+4+2+0+6+3+5)= 252 ... rw2
 24+38+27+37+31+33+28+34=  (3+4+3+4+3+4+3+4)*8+(0+6+3+5+7+1+4+2)= 252 ... rw3
 47+17+44+18+40+22+43+21=  (5+2+5+2+5+2+5+2)*8+(7+1+4+2+0+6+3+5)= 252 ... rw4
 56+6+59+5+63+1+60+2=      (7+0+7+0+7+0+7+0)*8+(0+6+3+5+7+1+4+2)= 252 ... rw5
 15+49+12+50+8+54+11+53=   (1+6+1+6+1+6+1+6)*8+(7+1+4+2+0+6+3+5)= 252 ... rw6
 32+30+35+29+39+25+36+26=  (4+3+4+3+4+3+4+3)*8+(0+6+3+5+7+1+4+2)= 252 ... rw7
 23+41+20+42+16+46+19+45=  (2+5+2+5+2+5+2+5)*8+(7+1+4+2+0+6+3+5)= 252 ... rw8
```

```
0+55+24+47+56+15+32+23=  (0+6+3+5+7+1+4+2)*8+(0+7+0+7+0+7+0+7)= 252 ... cl1
62+9+38+17+6+49+30+41=   (7+1+4+2+0+6+3+5)*8+(6+1+6+1+6+1+6+1)= 252 ... cl2
3+52+27+44+59+12+35+20=  (0+6+3+5+7+1+4+2)*8+(3+4+3+4+3+4+3+4)= 252 ... cl3
61+10+37+18+5+50+29+42=  (7+1+4+2+0+6+3+5)*8+(5+2+5+2+5+2+5+2)= 252 ... cl4
7+48+31+40+63+8+39+16=   (0+6+3+5+7+1+4+2)*8+(7+0+7+0+7+0+7+0)= 252 ... cl5
57+14+33+22+1+54+25+46=  (7+1+4+2+0+6+3+5)*8+(1+6+1+6+1+6+1+6)= 252 ... cl6
4+51+28+43+60+11+36+19=  (0+6+3+5+7+1+4+2)*8+(4+3+4+3+4+3+4+3)= 252 ... cl7
58+13+34+21+2+53+26+45=  (7+1+4+2+0+6+3+5)*8+(2+5+2+5+2+5+2+5)= 252 ... cl8
0+9+27+18+63+54+36+45=   (0+1+3+2+7+6+4+5)*8+(0+1+3+2+7+6+4+5)= 252 ... pd1
62+52+37+40+1+11+26+23=  (7+6+4+5+0+1+3+2)*8+(6+4+5+0+1+3+2+7)= 252 ... pd2

4+14+31+18+59+49+32+45=  (0+1+3+2+7+6+4+5)*8+(4+6+7+2+3+1+0+5)= 252 ... pb7
58+51+33+40+5+12+30+23=  (7+6+4+5+0+1+3+2)*8+(2+3+1+0+5+4+6+7)= 252 ... pb8
. . . . . .
```

** The same 'C & C' Magic Squares 8x8 with /D4i **

```
/N4i                                   /D4i       *16/             *4/              *1/
000 332 003 331 013 321 010 322   0 3 0 3 0 3 0 3   0 3 0 3 1 2 1 2   0 2 3 1 3 1 0 2
313 021 310 022 300 032 303 031   3 0 3 0 3 0 3 0   1 2 1 2 0 3 0 3   3 1 0 2 0 2 3 1
120 212 123 211 133 201 130 202   1 2 1 2 1 2 1 2   2 1 2 1 3 0 3 0   0 2 3 1 3 1 0 2
233 101 230 102 220 112 223 111   2 1 2 1 2 1 2 1   3 0 3 0 2 1 2 1   3 1 0 2 0 2 3 1
320 012 323 011 333 001 330 002   3 0 3 0 3 0 3 0   2 1 2 1 3 0 3 0   0 2 3 1 3 1 0 2
033 301 030 302 020 312 023 311   0 3 0 3 0 3 0 3   3 0 3 0 2 1 2 1   3 1 0 2 0 2 3 1
200 132 203 131 213 121 210 122   2 1 2 1 2 1 2 1   0 3 0 3 1 2 1 2   0 2 3 1 3 1 0 2
113 221 110 222 100 232 103 231   1 2 1 2 1 2 1 2   1 2 1 2 0 3 0 3   3 1 0 2 0 2 3 1
```

* Check Line-Sums *

```
0+62+3+61+7+57+4+58=      (0+3+0+3+0+3+0+3)*16+(0+3+0+3+1+2+1+2)*4+(0+2+3+1+3+1+0+2)=252 ... rw1
55+9+52+10+48+14+51+13=   (3+0+3+0+3+0+3+0)*16+(1+2+1+2+0+3+0+3)*4+(3+1+0+2+0+2+3+1)=252 ... rw2
24+38+27+37+31+33+28+34=  (1+2+1+2+1+2+1+2)*16+(2+1+2+1+3+0+3+0)*4+(0+2+3+1+3+1+0+2)=252 ... rw3

0+55+24+47+56+15+32+23=   (0+3+1+2+3+0+2+1)*16+(0+1+2+3+2+3+0+1)*4+(0+3+0+3+0+3+0+3)=252 ... cl1
62+9+38+17+6+49+30+41=    (3+0+2+1+0+3+1+2)*16+(3+2+1+0+1+0+3+2)*4+(2+1+2+1+2+1+2+1)=252 ... cl2
3+52+27+44+59+12+35+20=   (0+3+1+2+3+0+2+1)*16+(0+1+2+3+2+3+0+1)*4+(3+0+3+0+3+0+3+0)=252 ... cl3

0+9+27+18+63+54+36+45=    (0+0+1+1+3+3+2+2)*16+(0+2+2+0+3+1+1+3)*4+(0+1+3+2+3+2+0+1)=252 ... pd1
62+52+37+40+1+11+26+23=   (3+3+2+2+0+0+1+1)*16+(3+1+1+2+0+2+2+1)*4+(2+0+1+0+1+3+2+3)=252 ... pd2
3+10+31+22+60+53+32+41=   (0+0+1+1+3+3+2+2)*16+(0+2+3+1+3+1+0+2)*4+(3+2+3+2+0+1+0+1)=252 ... pd3

4+14+31+18+59+49+32+45=   (0+0+1+1+3+3+2+2)*16+(1+3+3+0+2+0+0+3)*4+(0+2+3+2+3+1+0+1)=252 ... pb7
58+51+33+40+5+12+30+23=   (3+3+2+2+0+0+1+1)*16+(2+0+0+2+1+3+3+1)*4+(2+3+1+0+1+0+2+3)=252 ... pb8
. . . . . .
```

** The same 'C & C' Magic Squares 8x8 with /D2i **

```
/N2i
000000 111110 000011 111101 000111 111001 000100 111010
110111 001001 110100 001010 110000 001110 110011 001101
011000 100110 011011 100101 011111 100001 011100 100010
101111 010001 101100 010010 101000 010110 101011 010101
111000 000110 111011 000101 111111 000001 111100 000010
001111 110001 001100 110010 001000 110110 001011 110101
100000 011110 100011 011101 100111 011001 100100 011010
010111 101001 010100 101010 010000 101110 010011 101101
```

```
/D2i*32/    *16/      *8/       *4/       *2/       *1/
01010101 01010101 01010101 01011010 01101001 00111100
10101010 10101010 01010101 10100101 10010110 11000011
01010101 10101010 10101010 01011010 01101001 00111100
10101010 01010101 10101010 10100101 10010110 11000011
10101010 10101010 10101010 01011010 01101001 00111100
01010101 01010101 10101010 10100101 10010110 11000011
10101010 01010101 01010101 01011010 01101001 00111100
01010101 10101010 01010101 10100101 10010110 11000011
```

*** Check Line-Sums ***

```
0+62+3+61+7+57+4+58=     (0+1+0+1+0+1+0+1)*32+(0+1+0+1+0+1+0+1)*16+(0+1+0+1+0+1+0+1)*8+
   (0+1+0+1+1+0+1+0)*4+(0+1+1+0+1+0+0+1)*2+(0+0+1+1+1+1+0+0)= 252 ... rw1
55+9+52+10+48+14+51+13=  (1+0+1+0+1+0+1+0)*32+(1+0+1+0+1+0+1+0)*16+(0+1+0+1+0+1+0+1)*8+
   (1+0+1+0+0+1+0+1)*4+(1+0+0+1+0+1+1+0)*2+(1+1+0+0+0+0+1+1)= 252 ... rw2
24+38+27+37+31+33+28+34= (0+1+0+1+0+1+0+1)*32+(1+0+1+0+1+0+1+0)*16+(1+0+1+0+1+0+1+0)*8+
   (0+1+0+1+1+0+1+0)*4+(0+1+1+0+1+0+0+1)*2+(0+0+1+1+1+1+0+0)= 252 ... rw3
47+17+44+18+40+22+43+21= (1+0+1+0+1+0+1+0)*32+(0+1+0+1+0+1+0+1)*16+(1+0+1+0+1+0+1+0)*8+
   (1+0+1+0+0+1+0+1)*4+(1+0+0+1+0+1+1+0)*2+(1+1+0+0+0+0+1+1)= 252 ... rw4
56+6+59+5+63+1+60+2=     (1+0+1+0+1+0+1+0)*32+(1+0+1+0+1+0+1+0)*16+(1+0+1+0+1+0+1+0)*8+
   (0+1+0+1+1+0+1+0)*4+(0+1+1+0+1+0+0+1)*2+(0+0+1+1+1+1+0+0)= 252 ... rw5

0+55+24+47+56+15+32+23=  (0+1+0+1+1+0+1+0)*32+(0+1+1+0+1+0+0+1)*16+(0+0+1+1+1+1+0+0)*8+
   (0+1+0+1+0+1+0+1)*4+(0+1+0+1+0+1+0+1)*2+(0+1+0+1+0+1+0+1)= 252 ... cl1
62+9+38+17+6+49+30+41=   (1+0+1+0+0+1+0+1)*32+(1+0+0+1+0+1+1+0)*16+(1+1+0+0+0+0+1+1)*8+
   (1+0+1+0+1+0+1+0)*4+(1+0+1+0+1+0+1+0)*2+(0+1+0+1+0+1+0+1)= 252 ... cl2
3+52+27+44+59+12+35+20=  (0+1+0+1+1+0+1+0)*32+(0+1+1+0+1+0+0+1)*16+(0+0+1+1+1+1+0+0)*8+
   (0+1+0+1+0+1+0+1)*4+(1+0+1+0+1+0+1+0)*2+(1+0+1+0+1+0+1+0)= 252 ... cl3
61+10+37+18+5+50+29+42=  (1+0+1+0+0+1+0+1)*32+(1+0+0+1+0+1+1+0)*16+(1+1+0+0+0+0+1+1)*8+
   (1+0+1+0+1+0+1+0)*4+(0+1+0+1+0+1+0+1)*2+(1+0+1+0+1+0+1+0)= 252 ... cl4
7+48+31+40+63+8+39+16=   (0+1+0+1+1+0+1+0)*32+(0+1+1+0+1+0+0+1)*16+(0+0+1+1+1+1+0+0)*8+
   (1+0+1+0+1+0+1+0)*4+(1+0+1+0+1+0+1+0)*2+(1+0+1+0+1+0+1+0)= 252 ... cl5

0+9+27+18+63+54+36+45=   (0+0+0+0+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+
   (0+0+0+0+1+1+1+1)*4+(0+0+1+1+1+1+0+0)*2+(0+1+1+0+1+0+0+1)= 252 ... pd1
62+52+37+40+1+11+26+23=  (1+1+1+1+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+
   (1+1+1+0+0+0+0+1)*4+(1+0+0+0+0+1+1+1)*2+(0+0+1+0+1+1+0+1)= 252 ... pd2
3+10+31+22+60+53+32+41=  (0+0+0+0+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+
   (0+0+1+1+1+1+0+0)*4+(1+1+1+1+0+0+0+0)*2+(1+0+1+0+0+1+0+1)= 252 ... pd3
61+48+33+43+2+15+30+20=  (1+1+1+1+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+
   (1+0+0+0+0+1+1+1)*4+(0+0+0+1+1+1+1+0)*2+(1+0+1+1+0+1+0+0)= 252 ... pd4
7+14+28+21+56+49+35+42=  (0+0+0+0+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+
   (1+1+1+1+0+0+0+0)*4+(1+1+0+0+0+0+1+1)*2+(1+0+0+1+0+1+1+0)= 252 ... pd5

7+10+27+17+56+53+36+46=  (0+0+0+0+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+
   (1+0+0+0+0+1+1+1)*4+(1+1+1+0+0+0+0+1)*2+(1+0+1+1+0+1+0+0)= 252 ... pb5
57+48+37+44+6+15+26+19=  (1+1+1+1+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+
   (0+0+1+1+1+1+0+0)*4+(0+0+0+0+1+1+1+1)*2+(1+0+1+0+0+1+0+1)= 252 ... pb6
4+14+31+18+59+49+32+45=  (0+0+0+0+1+1+1+1)*32+(0+0+1+1+1+1+0+0)*16+(0+1+1+0+1+0+0+1)*8+
   (1+1+1+0+0+0+0+1)*4+(0+1+1+1+1+0+0+0)*2+(0+0+1+0+1+1+0+1)= 252 ... pb7
58+51+33+40+5+12+30+23=  (1+1+1+1+0+0+0+0)*32+(1+1+0+0+0+0+1+1)*16+(1+0+0+1+0+1+1+0)*8+
   (0+0+0+0+1+1+1+1)*4+(1+1+0+0+0+0+1+1)*2+(0+1+1+0+1+0+0+1)= 252 ... pb8
 ... This is certainly a Pan-diagonal Magic Square!
0+45=45   ... cp1; 62+19=81 ... cp2; 3+46=49  ... cp3; 61+16=77 ... cp4;
7+42=49   ... cp5; 57+20=77 ... cp6; 4+41=45  ... cp7; 58+23=81 ... cp8;
55+26=81 ... cp9; 9+36=45   ... cp10;52+25=77 ... cp11;10+39=49 ... cp12;
48+29=77 ... cp13;14+35=49 ... cp14;51+30=81 ... cp15;13+32=45 ... cp16;
24+53=77 ... cp17;38+11=49 ... cp18;27+54=81 ... cp19;37+8=45  ... cp20;
31+50=81 ... cp21;33+12=45 ... cp22;28+49=77 ... cp23;34+15=49 ... cp24;
47+2=49   ... cp25;17+60=77 ... cp26;44+1=45  ... cp27;18+63=81 ... cp28;
40+5=45   ... cp29;22+59=81 ... cp30;43+6=49  ... cp31;21+56=77 ... cp32;
```

As you see, both /D8i and /D4i prove to be useless, for you cannot always find CES of order 8 by them. But only /D2i proves to be useful for you to do it.

We have to modify a little about the Euler's Conditions for order 8 as follows:

*** New Euler's Conditions for Order 8 ***

(1) Each line in each layer of /D2i forms must consist of {0, 0, 0, 0, 1, 1, 1, 1}. Number '0' must be used as often as '1' in any line. One must not be used more or less often than the other, nor must be un-used in any line.

(2) In the mathematical notation, any number of {0, 1, 2, 3, ..., 62, 63} must be used once. No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

```
  Vsum8= (0+0+0+0+1+1+1+1)*32+(0+0+0+0+1+1+1+1)*16+(0+0+0+0+1+1+1+1)*8+
     (0+0+0+0+1+1+1+1)*4+(0+0+0+0+1+1+1+1)*2+(0+0+0+0+1+1+1+1)= 252 ... (E8)
```

You don't have to care about the permutation of {0, 0, 0, 0, 1, 1, 1, 1} here. You only have to care for the only constant combination.

If you accept these modifications, you can surely find many Complete Euler Squares of order 8 among various types of pandiagonal MS88.

But how many Complete Euler Squares in all are there for order 8?
I know there are a lot of CES of order 8, but I have not yet succeeded in counting them through up to the last. I also know there are lots of "Non-Euler type" of pandiagonal magic squares of order 8, say, far more than CES.
What types of pandiagonal MS88 are all Complete Euler Squares, then?
Please take your kind look at the next list of 5 examples below.

**** Decomposition of Various Types of Pandiagonal Magic Squares 8x8 ****

```
 #1/Complete                  /D2i
   1 62  5 59  2 61 12 58  01010101 01010101 01010111 01100100 00010010 01001011
  57 14 50 48  9 18 45 19  10110010 10100101 11011010 01010010 00010001 01110100
  10 27 34 25 54 33 36 41  00101111 01011000 11010001 00001000 01000010 10101010
  49 30 26 23 52 21 22 37  10001001 11111110 01100000 01010111 00011000 01101010
  63  4 53  7 64  3 60  6  10101010 10101010 10001010 10111001 11011110 01001011
  56 47 20 46  8 51 15 17  11010100 10100101 01010010 11011010 11101110 10111000
  11 32 29 24 55 38 31 40  00001101 01111010 11100010 01111111 11011011 01010101
  13 44 43 28 16 35 39 42  01100111 00010000 11111001 10001010 01111110 01011001
 #2/Multiple, Complete        /D2i
   1 63  4 62 15 49 34 32  01010110 01010101 01011001 01011001 01101001 00110011
  60  7 54  9 41 39 20 30  10101100 10100011 10011001 01100101 11000110 10100011
  10 52 21 47 36 23 28 43  01011001 01100110 10010011 00110100 01011111 11001010
  59  8 51 12 38 19 48 25  10101010 10100101 10010011 01001010 11110110 01011010
  50 16 31 33 64  2 61  3  10011010 10101010 01101010 01101010 01101001 11001100
  24 26 45 35  5 58 11 56  00110101 11000101 01100110 10101001 10010011 11000101
  29 42 37 22 55 13 44 18  01101010 10011001 11000110 10111100 00001010 01010011
  27 46 17 40  6 57 14 53  01010101 10100101 11000110 01011011 10010000 01011010
 #3/'C & P'                   /D2i
   1 64  3 62  7 58  5 60  01010101 01010101 01010101 01011010 01101001 01010101
  63  2 61  4 57  8 59  6  10101010 10101010 10101010 10100101 10010110 01010101
  25 40 27 38 31 34 29 36  01010101 10101010 10101010 01011010 01101001 01010101
  39 26 37 28 33 32 35 30  10101010 01010101 01010101 10100101 10010110 01010101
  50 15 52 13 56  9 54 11  10101010 10101010 01010101 01011010 01101001 10101010
  16 49 14 51 10 55 12 53  01010101 01010101 10101010 10100101 10010110 10101010
  42 23 44 21 48 17 46 19  10101010 01010101 10101010 01011010 01101001 10101010
  24 41 22 43 18 47 20 45  01010101 10101010 01010101 10100101 10010110 10101010
 #4/'C & C'                   /D2i
   1 63  5 59 14 52 10 56  01010101 01010101 01011010 01101001 01010101 00001111
  62  4 58  8 49 15 53 11  10101010 10101010 10100101 10010110 01010101 11110000
  17 47 21 43 30 36 26 40  01010101 10101010 01011010 01101001 01010101 00001111
  46 20 42 24 33 31 37 27  10101010 01010101 10100101 10010110 01010101 11110000
  51 13 55  9 64  2 60  6  10101010 10101010 01011010 01101001 10101010 00001111
  16 50 12 54  3 61  7 57  01010101 01010101 10100101 10010110 10101010 11110000
  35 29 39 25 48 18 44 22  10101010 01010101 01011010 01101001 10101010 00001111
  32 34 28 38 19 45 23 41  01010101 10101010 10100101 10010110 10101010 11110000
```

```
 #5/Multiple,'C&C'          /D2i
  1 63  6 60 38 28 33 31   01011010 01010101 01010101 01101001 01010101 00111100
 62  4 57  7 25 39 30 36   10100101 10101010 10101010 10010110 01010101 11000011
 11 53 16 50 48 18 43 21   01011010 01010101 10101010 01101001 10101010 00111100
 56 10 51 13 19 45 24 42   10100101 10101010 01010101 10010110 10101010 11000011
 27 37 32 34 64  2 59  5   01011010 10101010 10101010 01101001 10101010 00111100
 40 26 35 29  3 61  8 58   10100101 01010101 01010101 10010110 10101010 11000011
 17 47 22 44 54 12 49 15   01011010 10101010 01010101 01101001 01010101 00111100
 46 20 41 23  9 55 14 52   10100101 01010101 10101010 10010110 01010101 11000011
```

The first two examples of complete MS88 are not Complete Euler Squares. You can find any other patterns than {0, 0, 0, 0, 1, 1, 1, 1} here and there in each /D2i layer.

The last three solutions above have no other patterns than {0, 0, 0, 0, 1, 1, 1, 1} in any line in each layer, even in all pandiagonals, as you see.

It seems 'Composite' type of pandiagonal MS88 are all Complete Euler Squares, whatever they might be even 'complete', 'multiple' or 'Simultaneous' at the same time.

#9. How about the Magic Squares of Order 9?

I would like you to see the next list of special example below. It is 'multiple' type of Simultaneous MS99: Self-complementary and Pandiagonal.

If you watch the /D9i diagrams and examine each number pattern in any line in any layer, you will easily know this is not a Complete Euler Square at all.

```
** 'Simultaneous' Magic Squares 9x9 with /D9i **
 1/Classic                    /Mathemat  .--.--.--.--.--.--.--.--.--.
  1 42 80 65 22 36 48 59 16   35 47 58 15| 0|41|79|64|21|35|47|58|15| 0 41 79 64
 50 61 12  6 44 73 67 27 29   72 66 26 28|49|60|11| 5|43|72|66|26|28|49 60 11  5
 72 20 31 52 57 14  8 37 78   13  7 36 77|71|19|30|51|56|13| 7|36|77|71 19 30 51
  7 39 77 71 19 33 54 56 13   32 53 55 12| 6|38|76|70|18|32|53|55|12| 6 38 76 70
 47 58 18  3 41 79 64 24 35   78 63 23 34|46|57|17| 2|40|78|63|23|34|46 57 17  2
 69 26 28 49 63 11  5 43 75   10  4 42 74|68|25|27|48|62|10| 4|42|74|68 25 27 48
  4 45 74 68 25 30 51 62 10   29 50 61  9| 3|44|73|67|24|29|50|61| 9| 3 44 73 67
 53 55 15  9 38 76 70 21 32   75 69 20 31|52|54|14| 8|37|75|69|20|31|52 54 14  8
 66 23 34 46 60 17  2 40 81   16  1 39 80|65|22|33|45|59|16| 1|39|80|65 22 33 45
                                         '--'--'--'--'--'--'--'--'--'
  /N9i                        /D9i          *9/                 *1/
 00 45 87 71 23 38 52 64 16   0 4 8 7 2 3 5 6 1   0 5 7 1 3 8 2 4 6
 54 66 12 05 47 80 73 28 31   5 6 1 0 4 8 7 2 3   4 6 2 5 7 0 3 8 1
 78 21 33 56 62 14 07 40 85   7 2 3 5 6 1 0 4 8   8 1 3 6 2 4 7 0 5
 06 42 84 77 20 35 58 61 13   0 4 8 7 2 3 5 6 1   6 2 4 7 0 5 8 1 3
 51 63 18 02 44 86 70 25 37   5 6 1 0 4 8 7 2 3   1 3 8 2 4 6 0 5 7
 75 27 30 53 68 11 04 46 82   7 2 3 5 6 1 0 4 8   5 7 0 3 8 1 4 6 2
 03 48 81 74 26 32 55 67 10   0 4 8 7 2 3 5 6 1   3 8 1 4 6 2 5 7 0
 57 60 15 08 41 83 76 22 34   5 6 1 0 4 8 7 2 3   7 0 5 8 1 3 6 2 4
 72 24 36 50 65 17 01 43 88   7 2 3 5 6 1 0 4 8   2 4 6 0 5 7 1 3 8
* Check Line-Sums *
 0+41+79+64+21+35+47+58+15= (0+4+8+7+2+3+5+6+1)*9+(0+5+7+1+3+8+2+4+6)= 360 ... rw1
 49+60+11+5+43+72+66+26+28= (5+6+1+0+4+8+7+2+3)*9+(4+6+2+5+7+0+3+8+1)= 360 ... rw2
 71+19+30+51+56+13+7+36+77= (7+2+3+5+6+1+0+4+8)*9+(8+1+3+6+2+4+7+0+5)= 360 ... rw3
 6+38+76+70+18+32+53+55+12= (0+4+8+7+2+3+5+6+1)*9+(6+2+4+7+0+5+8+1+3)= 360 ... rw4

 0+49+71+6+46+68+3+52+65=   (0+5+7+0+5+7+0+5+7)*9+(0+4+8+6+1+5+3+7+2)= 360 ... cl1
 41+60+19+38+57+25+44+54+22=(4+6+2+4+6+2+4+6+2)*9+(5+6+1+2+3+7+8+0+4)= 360 ... cl2
 79+11+30+76+17+27+73+14+33=(8+1+3+8+1+3+8+1+3)*9+(7+2+3+4+8+0+1+5+6)= 360 ... cl3
 64+5+51+70+2+48+67+8+45=   (7+0+5+7+0+5+7+0+5)*9+(1+5+6+7+2+3+4+8+0)= 360 ... cl4
```

```
0+60+30+70+40+10+50+20+80= (0+6+3+7+4+1+5+2+8)*9+(0+6+3+7+4+1+5+2+8)= 360 ... pd1
41+11+51+18+78+4+61+31+65= (4+1+5+2+8+0+6+3+7)*9+(5+2+6+0+6+4+7+4+2)= 360 ... pd2
79+5+56+32+63+42+9+52+22=  (8+0+6+3+7+4+1+5+2)*9+(7+5+2+5+0+6+0+7+4)= 360 ... pd3
64+43+13+53+23+74+3+54+33= (7+4+1+5+2+8+0+6+3)*9+(1+7+4+8+5+2+3+0+6)= 360 ... pd4

35+43+51+76+57+68+9+20+1=  (3+4+5+8+6+7+1+2+0)*9+(8+7+6+4+3+5+0+2+1)= 360 ... pb6
47+72+56+70+17+25+3+31+39= (5+8+6+7+1+2+0+3+4)*9+(2+0+2+7+8+7+3+4+3)= 360 ... pb7
58+66+13+18+2+27+44+52+80= (6+7+1+2+0+3+4+5+8)*9+(4+3+4+0+2+0+8+7+8)= 360 ... pb8
15+26+7+32+40+48+73+54+65= (1+2+0+3+4+5+8+6+7)*9+(6+8+7+5+4+3+1+0+2)= 360 ... pb9
```

**** The same 'Simultaneous' Magic Squares 9x9 with /D3i ****

```
/N3i                                          /D3i *27/      *9/       *3/       *1/
0000 1112 2221 2101 0210 1022 1202 2011 0120   012201120 012120201 012012012 021102210
1211 2020 0102 0012 1121 2200 2110 0222 1001   120012201 201012120 120120120 102210021
2122 0201 1010 1220 2002 0111 0021 1100 2212   201120012 120201012 201201201 210021102
0020 1102 2211 2121 0200 1012 1222 2001 0110   012201120 012120201 201201201 021102210
1201 2010 0122 0002 1111 2220 2100 0212 1021   120012201 201012120 012012012 102210021
2112 0221 1000 1210 2022 0101 0011 1120 2202   201120012 120201012 120120120 210021102
0010 1122 2201 2111 0220 1002 1212 2021 0100   012201120 012120201 120120120 021102210
1221 2000 0112 0022 1101 2210 2120 0202 1011   120012201 201012120 201201201 102210021
2102 0211 1020 1200 2012 0121 0001 1110 2222   201120012 120201012 012012012 210021102
```

*** Check Line-Sums ***

```
0+41+79+64+21+35+47+58+15= (0+1+2+2+0+1+1+2+0)*27+(0+1+2+1+2+0+2+0+1)*9+
  (0+1+2+0+1+2+0+1+2)*3+(0+2+1+1+0+2+2+1+0)= 360 ... rw1
49+60+11+5+43+72+66+26+28= (1+2+0+0+1+2+2+0+1)*27+(2+0+1+0+1+2+1+2+0)*9+
  (1+2+0+1+2+0+1+2+0)*3+(1+0+2+2+1+0+0+2+1)= 360 ... rw2
71+19+30+51+56+13+7+36+77= (2+0+1+1+2+0+0+1+2)*27+(1+2+0+2+0+1+0+1+2)*9+
  (2+0+1+2+0+1+2+0+1)*3+(2+1+0+0+2+1+1+0+2)= 360 ... rw3
6+38+76+70+18+32+53+55+12= (0+1+2+2+0+1+1+2+0)*27+(0+1+2+1+2+0+2+0+1)*9+
  (2+0+1+2+0+1+2+0+1)*3+(0+2+1+1+0+2+2+1+0)= 360 ... rw4

0+49+71+6+46+68+3+52+65= (0+1+2+0+1+2+0+1+2)*27+(0+2+1+0+2+1+0+2+1)*9+
  (0+1+2+2+0+1+1+2+0)*3+(0+1+2+0+1+2+0+1+2)= 360 ... cl1
41+60+19+38+57+25+44+54+22=(1+2+0+1+2+0+1+2+0)*27+(1+0+2+1+0+2+1+0+2)*9+
  (1+2+0+0+1+2+2+0+1)*3+(2+0+1+2+0+1+2+0+1)= 360 ... cl2
79+11+30+76+17+27+73+14+33=(2+0+1+2+0+1+2+0+1)*27+(2+1+0+2+1+0+2+1+0)*9+
  (2+0+1+1+2+0+0+1+2)*3+(1+2+0+1+2+0+1+2+0)= 360 ... cl3
64+5+51+70+2+48+67+8+45= (2+0+1+2+0+1+2+0+1)*27+(1+0+2+1+0+2+1+0+2)*9+
  (0+1+2+2+0+1+1+2+0)*3+(1+2+0+1+2+0+1+2+0)= 360 ... cl4

0+60+30+70+40+10+50+20+80= (0+2+1+2+1+0+1+0+2)*27+(0+0+0+1+1+1+2+2+2)*9+
  (0+2+1+2+1+0+1+0+2)*3+(0+0+0+1+1+1+2+2+2)= 360 ... pd1
41+11+51+18+78+4+61+31+65= (1+0+1+0+2+0+2+1+2)*27+(1+1+2+2+2+0+0+0+1)*9+
  (1+0+2+0+2+1+2+1+0)*3+(2+2+0+0+0+1+1+1+2)= 360 ... pd2
79+5+56+32+63+42+9+52+22=  (2+0+2+1+2+1+0+1+0)*27+(2+0+0+0+1+1+1+2+2)*9+
  (2+1+0+1+0+2+0+2+1)*3+(1+2+2+2+0+0+0+1+1)= 360 ... pd3
64+43+13+53+23+74+3+54+33= (2+1+0+1+0+2+0+2+1)*27+(1+1+1+2+2+2+0+0+0)*9+
  (0+2+1+2+1+0+1+0+2)*3+(1+1+1+2+2+2+0+0+0)= 360 ... pd4

35+43+51+76+57+68+9+20+1=  (1+1+1+2+2+2+0+0+0)*27+(0+1+2+2+0+1+1+2+0)*9+
  (2+2+2+1+1+1+0+0+0)*3+(2+1+0+1+0+2+0+2+1)= 360 ... pb6
47+72+56+70+17+25+3+31+39= (1+2+2+2+0+0+0+1+1)*27+(2+2+0+1+1+2+0+0+1)*9+
  (0+0+0+2+2+2+1+1+1)*3+(2+0+2+1+2+1+0+1+0)= 360 ... pb7
58+66+13+18+2+27+44+52+80= (2+2+0+0+0+1+1+1+2)*27+(0+1+1+2+0+0+1+2+2)*9+
  (1+1+1+0+0+0+2+2+2)*3+(1+0+1+0+2+0+2+1+2)= 360 ... pb8
15+26+7+32+40+48+73+54+65= (0+0+0+1+1+1+2+2+2)*27+(1+2+0+0+1+2+2+0+1)*9+
  (2+2+2+1+1+1+0+0+0)*3+(0+2+1+2+1+0+1+0+2)= 360 ... pb9
```

... This is certainly a Pan-diagonal Magic Square!

```
0+80=80  ... cp1;  41+39=80 ... cp2;  79+1=80  ... cp3;  64+16=80 ... cp4;
21+59=80 ... cp5;  35+45=80 ... cp6;  47+33=80 ... cp7;  58+22=80 ... cp8;
15+65=80 ... cp9;  49+31=80 ... cp10; 60+20=80 ... cp11; 11+69=80 ... cp12;
5+75=80  ... cp13; 43+37=80 ... cp14; 72+8=80  ... cp15; 66+14=80 ... cp16;
```

```
26+54=80 ... cp17;28+52=80 ... cp18;71+9=80  ... cp19;19+61=80 ... cp20;
30+50=80 ... cp21;51+29=80 ... cp22;56+24=80 ... cp23;13+67=80 ... cp24;
7+73=80  ... cp25;36+44=80 ... cp26;77+3=80  ... cp27;6+74=80  ... cp28;
38+42=80 ... cp29;76+4=80  ... cp30;70+10=80 ... cp31;18+62=80 ... cp32;
32+48=80 ... cp33;53+27=80 ... cp34;55+25=80 ... cp35;12+68=80 ... cp36;
46+34=80 ... cp37;57+23=80 ... cp38;17+63=80 ... cp39;2+78=80  ... cp40;
40+40=80 ... cp41;
 ... This is certainly a Self-complementary type!
```

But if you watch the /D3i diagrams of the second list of the same solution above and examine each number pattern in any line in any layer, you will surely find it is certainly a Complete Euler Square. It is the decompositions by the base 3 that you must have and draw the diagrams by. The more CES if you want to find, the better base of decompositions you should have.

Let's modify a little about the Euler's Conditions for order 9 as follows:

* New Euler's Conditions for Order 9 *

(1) Each line in each layer of /D3i forms must consist of {0, 0, 0, 1, 1, 1, 2, 2, 2}.
Any number of {0, 1 and 2} must be used as often as the others. One must not be used more or less often than the others, nor must be un-used in any line.

(2) In the mathematical notation, any number of {0, 1, 2, 3, ..., 79, 80} must be used once. No number must be used twice or more often, nor must be unused.

(3) Each line-sum must be expressed in the same form with the next equation:

```
Vsum9= (0+0+0+1+1+1+2+2+2)*27+(0+0+0+1+1+1+2+2+2)*9+
   (0+0+0+1+1+1+2+2+2)*3+(0+0+0+1+1+1+2+2+2)= 360 ... (E9)
```

You don't have to care about the permutation of {0, 0, 0, 1, 1, 1, 2, 2, 2} here. You only have to care for the only constant combination.

If you accept these modifications, you can surely find many Complete Euler Squares of order 9 among various types of pandiagonal MS99.

But how many Complete Euler Squares of order 9 are there in all?

I know there are not so many CES of order 9 as uncountable, but I have not yet counted them through. I am afraid it may take a very long time to do that.

To tell the truth, I had always failed in finding CES of order 9 for a long time. I had wondered if there might be no CES of order 9, just before I found some real samples of CES among the multiple type of pandiagonal ones including 9 little squares of order 3 within. I could find only a small number of solutions of that type, then.

It is getting really hard to find any CES of higher orders than 6. The higher orders, the fewer CES you can find. Those of higher orders are indeed rare, precious jewels.

I know nothing about how many "Non-Euler type" of MS99 exist. I am not interested in counting them through, because it seems to be practically impossible.

#10. Comment

What does the existence of 'Complete Euler Squares' mean?

I know it is a difficult question for us to answer, but the Complete Euler Squares seemed to be the common, essential structures of every pandiagonal magic squares of lower orders. They also seemed to mean the fundamental structures of all pan-diagonal magic squares of higher orders. But, were those impressions really true?

They really seemed to be true just as Legendary L. Euler had wanted them to be.

But another idea came up to me and has been growing up these days.

Complete Euler Squares may not always mean the essential structure of all pan-diagonal magic squares in general, but they may really mean the special key frames of composing the rare, precious jewels of special kinds.

All pandiagonal MS44 and MS55 are certainly Complete Euler Squares. But it is also true they are those rare, precious jewels of special kinds at the same time.
Miracle might have existed all the time from the beginning of magic squares.
But, I should only say now that we have to study more about them and discuss precisely with a lot of data about what they essentially mean in general.

At the end of this section, I want to tell you about my deepest appreciation to Dr. Mutsumi Suzuki. He has given me a lot of advice and inspiration to study magic squares with these modern mathematical arts, since I had never been fond of new notations and made myself stick to the classical style only.
"Without having any knowledge of those, you might surely miss any of the most important things. You will not be able to have the master keys to open the doors. Study 'Latin Squares', and learn some new notations at first," he once told me.
Now I would like to send this same message to you all beginners.

(Written on October 29, 2005 by Kanji Setsuda working on MacOSX and Xcode)

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