Chapter 2. New Study of Magic Squares 4x4: **Kanji Setsuda**

Section 3. Algebraic Study of 'Composite' Magic Squares 4x4

#1. What is the 'Composite' Type?

Suppose you pick up any 2 x 2 adjacent entries in the square, and add them up. When you always get any block-sum equal to the magic constant 34, then you may call 'Composite' for this type.

```
 1 12  6 15      1+12+16+ 5=34;   12+ 6+ 5+11=34;    6+15+11+ 2=34;
16  5 11  2     16+ 5+ 3+10=34;    5+11+10+ 8=34;   11+ 2+ 8+13=34;
 3 10  8 13      3+10+ 8+13=34;   10+ 8+ 7+ 9=34;    8+13+ 9+ 4=34;
14  7  9  4      . . . . . .
```

This name was once given by Prof. Mutsumi Suzuki in his page with his beautiful picture of Japanese royal flower. I know there are other names such as 'Most-perfect' or 'Compact' type, but I like the name 'Composite' best of all.

Let's study this type of magic squares right now. I like to start simply with the next Basic Form and the simultaneous 'Composite Conditions'.

[Figure 1: Basic Form and 'Composite' Conditions]

```
 .-----------.   n1+ n2+ n5+ n6=S  ....(1)
 |n1|n2|n3|n4|   n2+ n3+ n6+ n7=S  ....(2)
 |--+--+--+--|   n3+ n4+ n7+ n8=S  ....(3)
 |n5|n6|n7|n8|   n5+ n6+ n9+n10=S  ....(4)
 |--+--+--+--|   n6+ n7+n10+n11=S  ....(5)
 |n9|10|11|12|   n7+ n8+n11+n12=S  ....(6)
 |--+--+--+--|   n9+n10+n13+n14=S  ....(7)
 |13|14|15|16|  n10+n11+n14+n15=S  ....(8)
 '-----------'  n11+n12+n15+n16=S  ....(9)
```

If you want to make any magic squares of order 4 for this case, then you have to add the next three equations to the nine 'Composite' Conditions above:

```
  n1+n2+n3+n4=C    ...(10);   n1+n5+n9+n13=C   ...(11);
  n1+n6+n11+n16=C  ...(12);   [C means the Magic Constant 34]
```

First of all, let's make some algebraic study for this type only under those nine 'Composite Conditions'.

#2. Important Properties of 'Composite' Squares

What properties can we find only under those 9 Composite Conditions above? Let's compare them and calculate some.

```
 (1)+(3)+(7)+(9)
 n1+n2+n5+n6+n3+n4+n7+n8+n9+n10+n13+n14+n11+n12+n15+n16=4*S
 n1+n2+n3+n4+n5+n6+n7+n8+n9+n10+n11+n12+n13+n14+n15+n16=4*S
         (16+1)*16/2   =4*S
 Therefore            S=34=C          ....(0)

   n1+ n2+ n5+ n6=S....(1)
   n2+ n3+ n6+ n7=S....(2) --> n1+ n5=n3+ n7 (=P1)
   n3+ n4+ n7+ n8=S....(3) --> n2+ n6=n4+ n8 (=P2)
 (n1+n5)+(n2+n6)+(n3+n7)+(n4+n8)=P1+P2+P1+P2
   =(n1+n2+n5+n6)+(n3+n4+n7+n8)=2*S
```

```
Therefore P1+P2=S
As a consequence n1+n5+n4+n8=P1+P2=S ...(10)
On the other hand,
  n5+ n6+ n9+n10=S....(4)
  n6+ n7+n10+n11=S....(5) --> n5+ n9=n7+n11 (=P3)
  n7+ n8+n11+n12=S....(6) --> n6+n10=n8+n12 (=P4)
(n5+n9)+(n6+n10)+(n7+n11)+(n8+n12)=P3+P4+P3+P4
  =(n5+n6+n9+n10)+(n7+n8+n11+n12)=2*S
Therefore P3+P4=S
Consequently n5+n9+n8+n12=P3+P4=S   ...(11)
On the other hand,
  n9+n10+n13+n14=S  ....(7)
  n10+n11+n14+n15=S ....(8) -->   n9+n13=n11+n15 (=P5)
  n11+n12+n15+n16=S ....(9) -->  n10+n14=n12+n16 (=P6)
(n9+n13)+(n10+n14)+(n11+n15)+(n12+n16)=P5+P6+P5+P6
  =(n9+n10+n13+n14)+(n11+n12+n15+n16)=2*S
Therefore P5+P6=S
Consequently n9+n13+n12+n16=P5+P6=S ...(12)
```

```
    .-----------.
 n4|n1|n2|n3|n4|n1
   |--+--+--+--|
 n8|n5|n6|n7|n8|n5
   |--+--+--+--|
 12|n9|10|11|12|n9
   |--+--+--+--|
 16|13|14|15|16|13
   '-----------'
 n4 n1 n2 n3 n4 n1
```

(10), (11) and (12) tell us we could move the 1st column on the left far to the right beyond the 4th column. Or you could even move the 4th column on the right far to the left edge beyond the 1st column. Because any new 2x2 square accepts 'compoiste' conditions (10)~(12).

This is quite similar to one of the most important properties of pan-diagonal magic square, you know.

In the same way,

```
  n1+ n2+ n5+ n6=S...(1)
  n5+ n6+ n9+n10=S...(4) --> n1+ n2=n9+n10  (=Q1)
  n9+n10+n13+n14=S...(7) --> n5+ n6=n13+n14 (=Q2)
(n1+n2)+(n5+n6)+(n9+n10)+(n13+n14)=Q1+Q2+Q1+Q2
  =(n1+n2+n5+n6)+(n9+n10+n13+n14)=2*S
Therefore Q1+Q2=S
As a consequence n1+n2+n13+n14=Q1+Q2=S  ...(13)
  n2+n3=n10+n11 (=Q3); n6+n7=n14+n15 (=Q4)
  Q3+Q4=S; n2+n3+n14+n15=S              ...(14)
  n3+n4=n11+n12 (=Q5); n7+n8=n15+n16 (=Q6)
  Q5+Q6=S; n3+n4+n15+n16=S              ...(15)
```

(13), (14) and (15) show us we could move the 1st row far to the bottom below the 4th row. You could even move the 4th row far to the top above the 1st row.

This property is also similar to the one of pan-diagonal magic squares.

What relationship can we find among {n1, n4, n13 and n16}?

```
  n13+n14=n5+n6=Q2
Add (n1+n2) to both sides above.
n1+n2+n13+n14=n1+n2+n5+n6=S  ...(R1)
  n14+n15=n6+n7=Q4
Add (n2+n3) to both sides above.
n2+n3+n14+n15=n2+n3+n6+n7=S  ...(R2)
  n15+n16=n7+n8=Q6
Add (n3+n4) to both sides above.
```

```
  n3+n4+n15+n16=n3+n4+n7+n8=S  ...(R3)
(R1)+(R3)-(R2)
      n1+n2+n13+n14=S
  +)  n3+n4+n15+n16=S
  -)  n2+n3+n14+n15=S
----------------------------
     n1+n4+n13+n16=S           ...(16)
```

#3. Cross-Sums of 4 Tops of Inner 3 x 3 Little Squares

How about {n1, n3, n9, n11} or {n2, n4, n10, n12}?

```
   n1+ n5=n3+ n7=P1      |    n2+ n6=n4+ n8=P2
-) n5+ n9=n7+n11=P3      | -) n6+n10=n8+n12=P4
----------------------   | ----------------------
    n1-n9=n3-n11         |    n2-n10=n4-n12
   n1+n11=n3+n9....(CS1)|    n2+n12=n4+n10....(CS2)

   n5+ n9= n7+n11=P3     |    n6+n10=n8+n12=P4
-) n9+n13=n11+n15=P5     | -)n10+n14=n12+n16=P6
----------------------   | ----------------------
   n5-n13=n7-n15         |    n6-n14=n8-n16
   n5+n15=n7+n13...(CS3)|    n6+n16=n8+n14....(CS4)
```

Any cross-sum proves to be equal to each other.

#4. What about Pan-Diagonals?

What could we find about each sum of pandiagonal only under the nine 'Composite Conditions'?

[Figure 2: Extended Space and Pan-Diagonals]

```
 15 16 13 14 15 16 13 14 15  Pan-Diagonals:
      .-----------. -.        n1+ n6+n11+n16 = PD1
 n3 n4|n1|n2|n3|n4|n1|n2 n3   n1+ n8+n11+n14 = PD2
      |--+--+--+--| -|        n2+ n7+n12+n13 = PD3
 n7 n8|n5|n6|n7|n8|n5|n6 n7   n2+ n5+n12+n15 = PD4
      |--+--+--+--| -|        n3+ n8+ n9+n14 = PD5
 11 12|n9|10|11|12|n9|10 11   n3+ n6+ n9+n16 = PD6
      |--+--+--+--| -|        n4+ n5+n10+n15 = PD7
 15 16|13|14|15|16|13|14 15   n4+ n7+n10+n13 = PD8
      '-----------' -'
 n3 n4 n1 n2 n3 n4 n1 n2 n3
```

```
PD1=(n1+n11)+(n6+n16)     | PD3=(n2+n12)+(n7+n13)
PD6=(n3+n9)+(n6+n16)      | PD8=(n4+n10)+(n7+n13)
But n1+n11=n3+n9 ...(CS1)| But n2+n12=n4+n10 ...(CS2)
Therefore PD1=PD6         | Therefore PD3=PD8

PD2=(n1+n11)+(n8+n14)     | PD4=(n2+n12)+(n5+n15)
PD5=(n3+n9)+(n8+n14)      | PD7=(n4+n10)+(n5+n15)
But n1+n11=n3+n9 ...(CS1)| But n2+n12=n4+n10 ...(CS2)
Therefore PD2=PD5         | Therefore PD4=PD7

PD1=(n1+n11)+(n6+n16)     | PD3=(n2+n12)+(n7+n13)
PD2=(n1+n11)+(n8+n14)     | PD4=(n2+n12)+(n5+n15)
But n6+n16=n8+n14...(CS4)| But n5+n15=n7+n13 ...(CS3)
Therefore PD1=PD2         | Therefore PD3=PD4
```

```
PD5=(n3+n9)+(n8+n14)      | PD7=(n4+n10)+(n5+n15)
PD6=(n3+n9)+(n6+n16)      | PD8=(n4+n10)+(n7+n13)
But n6+n16=n8+n14...(CS4)| But n5+n15=n7+n13...(CS3)
Therefore PD5=PD6         | Therefore PD7=PD8

PD1=(n1+n11)+(n6+n16)     | PD2=(n1+n11)+(n8+n14)
PD5=(n3+n9)+(n8+n14)      | PD6=(n3+n9)+(n6+n16)
But n1+n11=n3+n9 ...(CS1)| But n1+n11=n3+n9 ...(CS1)
    n6+n16=n8+n14...(CS4)|     n6+n16=n8+n14...(CS4)
Therefore PD1=PD5         | Therefore PD2=PD6

PD3=(n2+n12)+(n7+n13)     | PD4=(n2+n12)+(n5+n15)
PD7=(n4+n10)+(n5+n15)     | PD8=(n4+n10)+(n7+n13)
But n2+n12=n4+n10...(CS2)| But n2+n12=n4+n10...(CS2)
    n5+n15=n7+n13...(CS3)|     n5+n15=n7+n13...(CS3)
Therefore PD3=PD7         | Therefore PD4=PD8
```

As a result PD1=PD2=PD5=PD6=P; PD3=PD4=PD7=PD8=Q; and P+Q=2S=2C=68
All pandiagonals can be classified into two groups according to their line-sums.
This property is almost the same as 'pan-magic', though PD1=PD8 is not yet always true only under the 9 Composite Conditions.
If you want PD1=PD8 or all pandiagonals equal to 34, then you must give PD1=PD8 or just 'PD1=34' to the basic conditions at the first definition stage.

#5. What about Row-sum or Column-sum?

Does any row or any column add up to the magic constant?

```
    n1+n2+n5+n6=S
+)  n3+n4+n7+n8=S
------------------------------
(n1+n2+n3+n4)+(n5+n6+n7+n8)=2*S

    n1+n2+n5+n6=S
+)  n9+n10+n13+n14=S
------------------------------
(n1+n5+n9+n13)+(n2+n6+n10+n14)=2*S
....
n1+n2+n3+n4=n9+n10+n11+n12
n1+n5+n9+n13=n3+n7+n11+n15
....
```

All rows can be classified into two groups according to their sums. All columns can also be classified into two groups according to their sums. But you cannot say they are all equal to 34 yet only under the nine Composite Conditions, although our 'Composite' type is almost the same as the 'Pandiagonal' magic type.
You may well call 'Semi-Panmagic' type for this 'Composite' one.
If you want all rows and columns be equal to the magic constant, then you must add both n1+n2+n3+n4=34 and n1+n5+n9+n13=34 to the list of Basic Conditions at the first definition stage.

#6. Making the 'Composite and Complete' Magic Square

Let's actually make a pandiagonal magic square of composite type now.
At first we only give the 9 Composite Conditions above and 3 equations as follows:

[Figure 1(Again): **Basic Conditions**]

```
    .-----------.   n1+ n2+ n5+ n6=S ....(1)
    |n1|n2|n3|n4|   n2+ n3+ n6+ n7=S ....(2)
    |--+--+--+--|   n3+ n4+ n7+ n8=S ....(3)
    |n5|n6|n7|n8|   n5+ n6+ n9+n10=S ....(4)
    |--+--+--+--|   n6+ n7+n10+n11=S ....(5)
    |n9|10|11|12|   n7+ n8+n11+n12=S ....(6)
    |--+--+--+--|   n9+n10+n13+n14=S ....(7)
    |13|14|15|16|  n10+n11+n14+n15=S ....(8)
    '-----------'  n11+n12+n15+n16=S ....(9)
   n1+n2+n3+n4=C      ...(10);    n1+n5+n9+n13=C    ...(11);
   n1+n6+n11+n16=C    ...(12)     (S=C=34)
```

Put {n1<n16; n1<n4; n1<n13; and n2<n5;} to prevent you from counting the same solution double just for any simple rotation or reflection of the pattern. Take those inequality conditions, and you will have the list of standard solutions.

Let me show you a sample list of computer program I wrote recently as follows.

```
/** 'Composite and Pandiagonal' Magic Squares 4x4 **/
/* 'MS44Cmpst.c' built by Kanji Setsuda */
/**  on Dec.18, 2004; Dec. 2, 2005; **/
/**  Worked on MacOSX and Xcode 1.5 **/
/**/
#include <stdio.h>
/**/
short cnt, cnt2;
short LSM;
short nm[17], uflg[17];
short anm[9][17];
/**/
void stp01(void), stp02(void), stp03(void), stp04(void);
void stp05(void), stp06(void), stp07(void), stp08(void);
void stp09(void), stp10(void), stp11(void), stp12(void);
void stp13(void), stp14(void), stp15(void), stp16(void);
void ansprint(void);
void printans(short x);
/**/
/* Main Program */
int main(){
 short n;
   printf("\n** 'Composite & Pandiagonal' Magic Squares 4x4 **\n");
   printf("***  Print the List of 48 Standard Solutions  ***\n");
   for(n=0;n<17;n++){nm[n]=0; uflg[n]=0;}
   LSM=34; cnt=0; cnt2=0;
   stp01();   /* Begin the Calculations */
   if(cnt2>0){printans(cnt2);}
   printf(" [Count = %d]\n",cnt);
   printf("  OK!\n");
 return 0;
}
/* Begin the Calculations */
/* Set n1 */
void stp01(){
 short a;
   for(a=1;a<17;a++){
      if(uflg[a]==0){
         nm[1]=a; uflg[a]=1;
```

```
        stp02();
        uflg[a]=0;}
   }
}
/* Set n2 */
void stp02(){
 short a;
   for(a=1;a<17;a++){
      if(uflg[a]==0){
        nm[2]=a; uflg[a]=1;
        stp03();
        uflg[a]=0;}
   }
}
/* Set n5 & n5>n2 */
void stp03(){
 short a;
   for(a=nm[2]+1;a<17;a++){
      if(uflg[a]==0){
        nm[5]=a; uflg[a]=1;
        stp04();
        uflg[a]=0;}
   }
}
/* Set n6=LSM-n1-n2-n5 */
void stp04(){
 short a;
   a=LSM-nm[1]-nm[2]-nm[5];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[6]=a; uflg[a]=1;
      stp05();
      uflg[a]=0;}
}
/* Set n3 */
void stp05(){
 short a;
   for(a=1;a<17;a++){
      if(uflg[a]==0){
        nm[3]=a; uflg[a]=1;
        stp06();
        uflg[a]=0;}
   }
}
/* Set n7=LSM-n2-n3-n6 */
void stp06(){
 short a;
   a=LSM-nm[2]-nm[3]-nm[6];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[7]=a; uflg[a]=1;
      stp07();
      uflg[a]=0;}
}
/* Set n4=LSM-n1-n2-n3 & n1<n4 */
void stp07(){
 short a;
   a=LSM-nm[1]-nm[2]-nm[3];
   if((nm[1]<a)&&(a<17)&&(uflg[a]==0)){
      nm[4]=a; uflg[a]=1;
      stp08();
```

```
      uflg[a]=0;}
}
/* Set n8=LSM-n3-n4-n7 */
void stp08(){
 short a;
   a=LSM-nm[3]-nm[4]-nm[7];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[8]=a; uflg[a]=1;
      stp09();
      uflg[a]=0;}
}
/* Set n9 */
void stp09(){
 short a;
   for(a=1;a<17;a++){
      if(uflg[a]==0){
         nm[9]=a; uflg[a]=1;
         stp10();
         uflg[a]=0;}
   }
}
/* Set n10=LSM-n5-n6-n9 */
void stp10(){
 short a;
   a=LSM-nm[5]-nm[6]-nm[9];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[10]=a; uflg[a]=1;
      stp11();
      uflg[a]=0;}
}
/* Set n11=LSM-n6-n7-n10 */
void stp11(){
 short a;
   a=LSM-nm[6]-nm[7]-nm[10];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[11]=a; uflg[a]=1;
      stp12();
      uflg[a]=0;}
}
/* Set n12=LSM-n7-n8-n11 */
void stp12(){
 short a;
   a=LSM-nm[7]-nm[8]-nm[11];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[12]=a; uflg[a]=1;
      stp13();
      uflg[a]=0;}
}
/* Set n13=LSM-n1-n5-n9 & n1<n13 */
void stp13(){
 short a;
   a=LSM-nm[1]-nm[5]-nm[9];
   if((nm[1]<a)&&(a<17)&&(uflg[a]==0)){
      nm[13]=a; uflg[a]=1;
      stp14();
      uflg[a]=0;}
}
/* Set n14=LSM-n9-n10-n13 */
void stp14(){
```

```
 short a;
   a=LSM-nm[9]-nm[10]-nm[13];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[14]=a; uflg[a]=1;
      stp15();
      uflg[a]=0;}
}
/* Set n15=LSM-n10-n11-n14 */
void stp15(){
 short a;
   a=LSM-nm[10]-nm[11]-nm[14];
   if((0<a)&&(a<17)&&(uflg[a]==0)){
      nm[15]=a; uflg[a]=1;
      stp16();
      uflg[a]=0;}
}
/* Set n16=LSM-n11-n12-n15 and */
/* Check n16=LSM-n1-n6-n11 & n1<n16 */
void stp16(){
 short a,b;
   a=LSM-nm[11]-nm[12]-nm[15];
   b=LSM-nm[1]-nm[6]-nm[11];
   if((nm[1]<a)&&(a<17)){
      if((a==b)&&(uflg[a]==0)){
         nm[16]=a; uflg[a]=1;
         ansprint();
         uflg[a]=0;}}
}
/**/
/* Print the Answers */
void ansprint(){
 short n;
   cnt++;
   anm[cnt2][0]=cnt;
   for(n=1;n<17;n++){anm[cnt2][n]=nm[n];}
   cnt2++; if(cnt2==6){printans(6); cnt2=0;}
}
/* Print the Answers */
void printans(short x){
 short m,n;
   for(m=0;m<x;m++){printf("%12d/",anm[m][0]);}
   printf("\n");
   for(m=0;m<x;m++){
      printf("%4d%3d%3d%3d",anm[m][1],anm[m][2],anm[m][3],anm[m][4]);}
   printf("\n");
   for(m=0;m<x;m++){
      printf("%4d%3d%3d%3d",anm[m][5],anm[m][6],anm[m][7],anm[m][8]);}
   printf("\n");
   for(m=0;m<x;m++){
      printf("%4d%3d%3d%3d",anm[m][9],anm[m][10],anm[m][11],anm[m][12]);}
   printf("\n");
   for(m=0;m<x;m++){
      printf("%4d%3d%3d%3d",anm[m][13],anm[m][14],anm[m][15],anm[m][16]);}
   printf("\n");
}
/**/
```

With this program you will surely get the next 48 solutions as follows.

```
** 'Composite & Pandiagonal' Magic Squares 4x4 **
***  Print the List of 48 Standard Solutions  ***
           1/           2/           3/           4/           5/           6/
  1  8 10 15   1  8 11 14   1  8 10 15   1  8 13 12   1  8 11 14   1  8 13 12
 12 13  3  6  12 13  2  7  14 11  5  4  14 11  2  7  15 10  5  4  15 10  3  6
  7  2 16  9   6  3 16  9   7  2 16  9   4  5 16  9   6  3 16  9   4  5 16  9
 14 11  5  4  15 10  5  4  12 13  3  6  15 10  3  6  12 13  2  7  14 11  2  7
           7/           8/           9/          10/          11/          12/
  1 12  6 15   1 12 13  8   1 12  7 14   1 12 13  8   1 14  7 12   1 14 11  8
 14  7  9  4  14  7  2 11  15  6  9  4  15  6  3 10  15  4  9  6  15  4  5 10
 11  2 16  5   4  9 16  5  10  3 16  5   4  9 16  5  10  5 16  3   6  9 16  3
  8 13  3 10  15  6  3 10   8 13  2 11  14  7  2 11   8 11  2 13  12  7  2 13
          13/          14/          15/          16/          17/          18/
  2  7  9 16   2  7 12 13   2  7  9 16   2  7 14 11   2  7 12 13   2  7 14 11
 11 14  4  5  11 14  1  8  13 12  6  3  13 12  1  8  16  9  6  3  16  9  4  5
  8  1 15 10   5  4 15 10   8  1 15 10   3  6 15 10   5  4 15 10   3  6 15 10
 13 12  6  3  16  9  6  3  11 14  4  5  16  9  4  5  11 14  1  8  13 12  1  8
          19/          20/          21/          22/          23/          24/
  2 11  5 16   2 11 14  7   2 11  8 13   2 11 14  7   2 13  8 11   2 13 12  7
 13  8 10  3  13  8  1 12  16  5 10  3  16  5  4  9  16  3 10  5  16  3  6  9
 12  1 15  6   3 10 15  6   9  4 15  6   3 10 15  6   9  6 15  4   5 10 15  4
  7 14  4  9  16  5  4  9   7 14  1 12  13  8  1 12   7 12  1 14  11  8  1 14
          25/          26/          27/          28/          29/          30/
  3  6  9 16   3  6 15 10   3  6 12 13   3  6 15 10   3 10  5 16   3 10 15  6
 13 12  7  2  13 12  1  8  16  9  7  2  16  9  4  5  13  8 11  2  13  8  1 12
  8  1 14 11   2  7 14 11   5  4 14 11   2  7 14 11  12  1 14  7   2 11 14  7
 10 15  4  5  16  9  4  5  10 15  1  8  13 12  1  8   6 15  4  9  16  5  4  9
          31/          32/          33/          34/          35/          36/
  3 10  8 13   3 10 15  6   3 13  8 10   3 13 12  6   4  5 10 15   4  5 16  9
 16  5 11  2  16  5  4  9  16  2 11  5  16  2  7  9  14 11  8  1  14 11  2  7
  9  4 14  7   2 11 14  7   9  7 14  4   5 11 14  4   7  2 13 12   1  8 13 12
  6 15  1 12  13  8  1 12   6 12  1 15  10  8  1 15   9 16  3  6  15 10  3  6
          37/          38/          39/          40/          41/          42/
  4  5 11 14   4  5 16  9   4  9  6 15   4  9 16  5   4  9  7 14   4  9 16  5
 15 10  8  1  15 10  3  6  14  7 12  1  14  7  2 11  15  6 12  1  15  6  3 10
  6  3 13 12   1  8 13 12  11  2 13  8   1 12 13  8  10  3 13  8   1 12 13  8
  9 16  2  7  14 11  2  7   5 16  3 10  15  6  3 10   5 16  2 11  14  7  2 11
          43/          44/          45/          46/          47/          48/
  4 14  7  9   4 14 11  5   5  4 14 11   5  4 15 10   6  3 13 12   6  3 16  9
 15  1 12  6  15  1  8 10  16  9  7  2  16  9  6  3  15 10  8  1  15 10  5  4
 10  8 13  3   6 12 13  3   3  6 12 13   2  7 12 13   4  5 11 14   1  8 11 14
  5 11  2 16   9  7  2 16  10 15  1  8  11 14  1  8   9 16  2  7  12 13  2  7
 [Count = 48]  OK!
```

As you see in the list above, any complementary pairs of 17: {(1,16), (2,15), (3,14), (4,13), (5,12), (6,11), (7,10), and (8,9)} are all placed on their pandiagonals in every solution. Yes. They are naturally the 'Complete' magic squares of order 4, as well as the 'Pan-diagonal' magic one in general.

Any of those three sets of 'Composite', 'Complete' and 'Pandiagonal' magic squares of order 4 is really the same with one another. It is amazing.

Every time when I could see this miraculous agreement, I used to be moved deep and also puzzled so much as to make myself try to find any reason for that.

#7. The 'Composite' Semi-Panmagic Squares of Order 4

Let's get back to the first step to the nine Composite Conditions, and examine whether we can make any 'Composite' semi-panmagic square of order 4 or not with 11 equations but without the last for any diagonal equal to the constant 34.

[Figure 1(Again): Basic Conditions]

```
  .-----------.    n1+ n2+ n5+ n6=S ....(1)
  |n1|n2|n3|n4|    n2+ n3+ n6+ n7=S ....(2)
  |--+--+--+--|    n3+ n4+ n7+ n8=S ....(3)
  |n5|n6|n7|n8|    n5+ n6+ n9+n10=S ....(4)
  |--+--+--+--|    n6+ n7+n10+n11=S ....(5)
  |n9|10|11|12|    n7+ n8+n11+n12=S ....(6)
  |--+--+--+--|    n9+n10+n13+n14=S ....(7)
  |13|14|15|16|   n10+n11+n14+n15=S ....(8)
  '-----------'   n11+n12+n15+n16=S ....(9)

  n1+n2+n3+n4=C  ...(10);   n1+n5+n9+n13=C ...(11); (S=C=34)
```

Unless we give n1+n6+n11+n16=34 at the definition stage, any pandiagonal does not always add up to the same constant 34.

But anyone with n1 or n3 takes the same sum P, and anyone with n2 or n4 takes the same sum Q. (P+Q=2*S=2*C=72) It is nearly almost a pan-magic square.

How many solutions in all can we find for them?

**** 'Composite' Semi-Panmagic Squares 4x4 ****

** List of the Primitive 1920 Solutions **

```
         1/          7/          13/          19/          25/
  1  8  9 16   1  8  9 16   1  8  9 16   1  8 10 15   1 12  5 16
 12 13  4  5  14 11  6  3  15 10  7  2  16  9  7  2   8 13  4  9
  6  3 14 11   4  5 12 13   4  5 12 13   3  6 12 13  10  3 14  7
 15 10  7  2  15 10  7  2  14 11  6  3  14 11  5  4  15  6 11  2
        31/          37/          43/          49/          55/
  1 12  5 16   1 12  5 16   1 12  6 15   1 14  3 16   1 14  3 16
 14  7 10  3  15  6 11  2  16  5 11  2   8 11  6  9  12  7 10  5
  4  9  8 13   4  9  8 13   3 10  8 13  10  5 12  7   6  9  8 11
 15  6 11  2  14  7 10  3  14  7  9  4  15  4 13  2  15  4 13  2
        61/          67/          73/          79/          85/
  1 14  3 16   1 14  4 15   1 15  2 16   1 15  2 16   1 15  2 16
 15  4 13  2  16  3 13  2   8 10  7  9  12  6 11  5  14  4 13  3
  6  9  8 11   5 10  8 11  11  5 12  6   7  9  8 10   7  9  8 10
 12  7 10  5  12  7  9  6  14  4 13  3  14  4 13  3  12  6 11  5
        91/          97/         103/         109/         115/
  1 15  4 14   1 16  2 15   1 16  2 15   1 16  2 15   1 16  3 14
 16  2 13  3   8  9  7 10  12  5 11  6  14  3 13  4  15  2 13  4
  5 11  8 10  11  6 12  5   7 10  8  9   7 10  8  9   6 11  8  9
 12  6  9  7  14  3 13  4  14  3 13  4  12  5 11  6  12  5 10  7
       121/         145/         169/         193/         217/
  2  7  9 16   2 11  5 16   2 13  3 16   2 15  1 16   2 16  1 15
 11 14  4  5   7 14  4  9   7 12  6  9   7 10  8  9   7  9  8 10
  6  3 13 12  10  3 13  8  10  5 11  8  12  5 11  6  12  6 11  5
 15 10  8  1  15  6 12  1  15  4 14  1  13  4 14  3  13  3 14  4
```

			241/				265/				289/				313/				337/
3	6	9	16	3	10	5	16	3	13	2	16	3	14	1	16	3	16	1	14
10	15	4	5	6	15	4	9	6	12	7	9	6	11	8	9	6	9	8	11
7	2	13	12	11	2	13	8	11	5	10	8	12	5	10	7	12	7	10	5
14	11	8	1	14	7	12	1	14	4	15	1	13	4	15	2	13	2	15	4

			361/				385/				409/				433/				457/
4	5	10	15	4	9	6	15	4	13	2	15	4	14	1	15	4	15	1	14
9	16	3	6	5	16	3	10	5	12	7	10	5	11	8	10	5	10	8	11
7	2	13	12	11	2	13	8	11	6	9	8	12	6	9	7	12	7	9	6
14	11	8	1	14	7	12	1	14	3	16	1	13	3	16	2	13	2	16	3

			481/				505/				529/				553/				577/
5	4	9	16	5	10	3	16	5	11	2	16	5	12	1	16	5	16	1	12
10	15	6	3	4	15	6	9	4	14	7	9	4	13	8	9	4	9	8	13
7	2	11	14	13	2	11	8	13	3	10	8	14	3	10	7	14	7	10	3
12	13	8	1	12	7	14	1	12	6	15	1	11	6	15	2	11	2	15	6

			601/				625/				649/				673/				697/
6	3	10	15	6	9	4	15	6	11	2	15	6	12	1	15	6	15	1	12
9	16	5	4	3	16	5	10	3	14	7	10	3	13	8	10	3	10	8	13
7	2	11	14	13	2	11	8	13	4	9	8	14	4	9	7	14	7	9	4
12	13	8	1	12	7	14	1	12	5	16	1	11	5	16	2	11	2	16	5

			721/				745/				769/				793/				817/
7	2	11	14	7	9	4	14	7	10	3	14	7	12	1	14	7	14	1	12
9	16	5	4	2	16	5	11	2	15	6	11	2	13	8	11	2	11	8	13
6	3	10	15	13	3	10	8	13	4	9	8	15	4	9	6	15	6	9	4
12	13	8	1	12	6	15	1	12	5	16	1	10	5	16	3	10	3	16	5

			841/				865/				889/				913/				937/
8	1	12	13	8	9	4	13	8	10	3	13	8	11	2	13	8	13	2	11
9	16	5	4	1	16	5	12	1	15	6	12	1	14	7	12	1	12	7	14
6	3	10	15	14	3	10	7	14	4	9	7	15	4	9	6	15	6	9	4
11	14	7	2	11	6	15	2	11	5	16	2	10	5	16	3	10	3	16	5

			961/				985/				1009/				1033/				1057/
9	4	5	16	9	6	3	16	9	7	2	16	9	8	1	16	9	16	1	8
6	15	10	3	4	15	10	5	4	14	11	5	4	13	12	5	4	5	12	13
11	2	7	14	13	2	7	12	13	3	6	12	14	3	6	11	14	11	6	3
8	13	12	1	8	11	14	1	8	10	15	1	7	10	15	2	7	2	15	10

			1081/				1105/				1129/				1153/				1177/
10	3	6	15	10	5	4	15	10	7	2	15	10	8	1	15	10	15	1	8
5	16	9	4	3	16	9	6	3	14	11	6	3	13	12	6	3	6	12	13
11	2	7	14	13	2	7	12	13	4	5	12	14	4	5	11	14	11	5	4
8	13	12	1	8	11	14	1	8	9	16	1	7	9	16	2	7	2	16	9

			1201/				1225/				1249/				1273/				1297/
11	2	7	14	11	5	4	14	11	6	3	14	11	8	1	14	11	14	1	8
5	16	9	4	2	16	9	7	2	15	10	7	2	13	12	7	2	7	12	13
10	3	6	15	13	3	6	12	13	4	5	12	15	4	5	10	15	10	5	4
8	13	12	1	8	10	15	1	8	9	16	1	6	9	16	3	6	3	16	9

			1321/				1345/				1369/				1393/				1417/
12	1	8	13	12	5	4	13	12	6	3	13	12	7	2	13	12	13	2	7
5	16	9	4	1	16	9	8	1	15	10	8	1	14	11	8	1	8	11	14
10	3	6	15	14	3	6	11	14	4	5	11	15	4	5	10	15	10	5	4
7	14	11	2	7	10	15	2	7	9	16	2	6	9	16	3	6	3	16	9

```
        1441/        1465/        1489/        1513/        1537/
13  2  7 12  13  3  6 12  13  4  5 12  13  8  1 12  13 12  1  8
 3 16  9  6   2 16  9  7   2 15 10  7   2 11 14  7   2  7 14 11
10  5  4 15  11  5  4 14  11  6  3 14  15  6  3 10  15 10  3  6
 8 11 14  1   8 10 15  1   8  9 16  1   4  9 16  5   4  5 16  9
        1561/        1585/        1609/        1633/        1657/
14  1  8 11  14  3  6 11  14  4  5 11  14  7  2 11  14 11  2  7
 3 16  9  6   1 16  9  8   1 15 10  8   1 12 13  8   1  8 13 12
10  5  4 15  12  5  4 13  12  6  3 13  15  6  3 10  15 10  3  6
 7 12 13  2   7 10 15  2   7  9 16  2   4  9 16  5   4  5 16  9
        1681/        1705/        1729/        1753/        1777/
15  1  8 10  15  2  7 10  15  4  5 10  15  6  3 10  15 10  3  6
 2 16  9  7   1 16  9  8   1 14 11  8   1 12 13  8   1  8 13 12
11  5  4 14  12  5  4 13  12  7  2 13  14  7  2 11  14 11  2  7
 6 12 13  3   6 11 14  3   6  9 16  3   4  9 16  5   4  5 16  9
        1801/        1825/        1849/        1873/        1897/
16  1  8  9  16  2  7  9  16  3  6  9  16  5  4  9  16  9  4  5
 2 15 10  7   1 15 10  8   1 14 11  8   1 12 13  8   1  8 13 12
11  6  3 14  12  6  3 13  12  7  2 13  14  7  2 11  14 11  2  7
 5 12 13  4   5 11 14  4   5 10 15  4   3 10 15  6   3  6 15 10
 [Count = 1920]

* Counts accordig to the Sum of Diagonals: *
  (14,54):[  0,  0];   (15,53):[  0,  0];   (16,52):[  0,  0];
  (17,51):[  0,  0];   (18,50):[192,192];   (19,49):[  0,  0];
  (20,48):[  0,  0];   (21,47):[  0,  0];   (22,46):[  0,  0];
  (23,45):[  0,  0];   (24,44):[  0,  0];   (25,43):[  0,  0];
  (26,42):[192,192];   (27,41):[  0,  0];   (28,40):[  0,  0];
  (29,39):[  0,  0];   (30,38):[192,192];   (31,37):[  0,  0];
  (32,36):[192,192];   (33,35):[  0,  0];   (34,34):[384,384];
  (35,33):[  0,  0];   (36,32):[192,192];   (37,31):[  0,  0];
  (38,30):[192,192];   (39,29):[  0,  0];   (40,28):[  0,  0];
  (41,27):[  0,  0];   (42,26):[192,192];   (43,25):[  0,  0];
  (44,24):[  0,  0];   (45,23):[  0,  0];   (46,22):[  0,  0];
  (47,21):[  0,  0];   (48,20):[  0,  0];   (49,19):[  0,  0];
  (50,18):[192,192];   (51,17):[  0,  0];   (52,16):[  0,  0];
  (53,15):[  0,  0];   (54,14):[  0,  0];   (55,13):[  0,  0];
```

If you take the basic conditions (1)~(11), you will find 1920 'primitive' solutions in all as shown above. This list contains the primitive 384 solutions of 'Composite and Complete' pan-magic squares in it. If you remove them, you will have the genuine 1536(=1920-384) 'Composite' and 'Semi-Panmagic' squares as a result.

#8. Can we assume the Extended Space and any Transformation System?

During our study in the previous #2~#4 we know we can assume the Extended Space for our Composite type in the same way as the pandiagonal case.

Since we can move any line in the first place far to the opposite side beyond the last place, we can draw such an Extended Form for our object as follows.

You will never lose any important properties of 'Composite Squares' anywhere in the Extended Space. Any 2x2 little block always adds up to the constant-sum 34, and any pandiagonal also adds up to the same sum of either P or Q.

But can we really read any other pieces of solutions out of the Extended Space by the moving frame, just in the same way as pandiagonal magic squares?

[Figure 2: Concept Diagrams for the **Extended Space**]

```
* Basic Form in Extended Space and the Moving Frame *
 15 16 13 14 15 16 13 14 15
      .--.--.--.--.                .--.--.--.--.--.
  3  4|n1|n2|n3|n4| 1  2  3      3  4|n1|n2|n3|n4| 1| 2  3
      |--+--+--+--|                  |--+--+--+--|--.
  7  8|n5|n6|n7|n8| 5  6  7      7  8|n5|n6|n7|n8| 5| 6  7
      |--+--+--+--|                  |--+--+--+--|--|
 11 12|n9|10|11|12| 9 10 11     11 12|n9|10|11|12| 9|10 11
      |--+--+--+--|                  |--+--+--+--|--|
 15 16|13|14|15|16|13 14 15     15 16|13|14|15|16|13|14 15
      '--'--'--'--'                  '--'--'--'--'--'
  3  4  1  2  3  4  1  2  3      3  4| 1| 2| 3| 4| 1| 2  3
                                     '--'--'--'--'--'
  7  8  5  6  7  8  5  6  7
```

Yes. We can read them out of the Extended Space above as follows. We can take out four different forms with each n1 on any four corners of the new squares.

```
* Read 4 Forms out of the Extended Space *
          T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 |n1|n2|n3|n4|  | 2| 3| 4| 1|  | 5| 6| 7| 8|  | 6| 7| 8| 5|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |n5|n6|n7|n8|  | 6| 7| 8| 5|  | 9|10|11|12|  |10|11|12| 9|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |n9|10|11|12|  |10|11|12| 9|  |13|14|15|16|  |14|15|16|13|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |13|14|15|16|  |14|15|16|13|  | 1| 2| 3| 4|  | 2| 3| 4| 1|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
* Make them Smart for n1=1 always on the left top *
          T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1| 2| 3| 4|  | 1| 4| 3| 2|  | 1| 2| 3| 4|  | 1| 4| 3| 2|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 5| 6| 7| 8|  | 5| 8| 7| 6|  |13|14|15|16|  |13|16|15|14|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 9|10|11|12|  | 9|12|11|10|  | 9|10|11|12|  | 9|12|11|10|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |13|14|15|16|  |13|16|15|14|  | 5| 6| 7| 8|  | 5| 8| 7| 6|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

Of course, you can make them smart by simple rotation and reflection as shown above so as to meet with n1<n4; n1<n13; n1<n16, and n2<n5.
Even if you do this job, you will never spoil any constant sum of 'Composite' equations, any rows and columns, and even any pandiagonals (equal to either P or Q, but not always equal to 34).

All those diagrams above imply the possibility of some transformation system of solutions. Suppose any solution is given to us. We can now transform it into 4 different forms automatically by the extended space and the moving frame.
We can also take the concept of 'fundamental' solution for a representative of the 4 solutions transformed.

The next diagrams will demonstrate the actual process of transformation.

* **Original #1** in Extended Form *   * **Moving Frame** *

```
  7  2 15 10  7  2 15 10  7
       .--.--.--.--.                .--.--.--.--.--.
  9 16| 1| 8| 9|16| 1  8  9     9 16| 1| 8| 9|16| 1| 8  9
      |--+--+--+--|                 |--+--+--+--|--.
  4  5|12|13| 4| 5|12 13  4     4  5|12|13| 4| 5|12|13  4
      |--+--+--+--|                 |--+--+--+--|--|
 14 11| 6| 3|14|11| 6  3 14    14 11| 6| 3|14|11| 6| 3 14
      |--+--+--+--|                 |--+--+--+--|--|
  7  2|15|10| 7| 2|15 10  7     7  2|15|10| 7| 2|15|10  7
       '--'--'--'--'                '--'--'--'--'--'
  9 16  1  8  9 16  1  8  9     9 16| 1| 8| 9|16| 1| 8  9
                                     '--'--'--'--'--'
```

* Read 4 Forms out of the Extended Space *

```
           T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1| 8| 9|16|  | 8| 9|16| 1|  |12|13| 4| 5|  |13| 4| 5|12|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |12|13| 4| 5|  |13| 4| 5|12|  | 6| 3|14|11|  | 3|14|11| 6|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 6| 3|14|11|  | 3|14|11| 6|  |15|10| 7| 2|  |10| 7| 2|15|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |15|10| 7| 2|  |10| 7| 2|15|  | 1| 8| 9|16|  | 8| 9|16| 1|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

* Make them Smart for n1=1 always on the left top *

```
           T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1| 8| 9|16|  | 1|12| 6|15|  | 1| 8| 9|16|  | 1|15| 6|12|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |12|13| 4| 5|  |16| 5|11| 2|  |15|10| 7| 2|  |16| 2|11| 5|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 6| 3|14|11|  | 9| 4|14| 7|  | 6| 3|14|11|  | 9| 7|14| 4|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |15|10| 7| 2|  | 8|13| 3|10|  |12|13| 4| 5|  | 8|10| 3|13|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

* **Original #2** in Extended Form *   * **Moving Frame** *

```
  7  2 11 14  7  2 11 14  7
       .--.--.--.--.                .--.--.--.--.--.
 12 13| 8| 1|12|13| 8  1 12    13  8| 1|12|13| 8| 1|12 13
      |--+--+--+--|                 |--+--+--+--|--.
  5  4| 9|16| 5| 4| 9 16  5     4  9|16| 5| 4| 9|16| 5  4
      |--+--+--+--|                 |--+--+--+--|--|
 10 15| 6| 3|10|15| 6  3 10    15  6| 3|10|15| 6| 3|10 15
      |--+--+--+--|                 |--+--+--+--|--|
  7  2|11|14| 7| 2|11 14  7     2 11|14| 7| 2|11|14| 7  2
       '--'--'--'--'                '--'--'--'--'--'
 12 13  8  1 12 13  8  1 12    13  8| 1|12|13| 8| 1|12 13
                                     '--'--'--'--'--'
```

* Read 4 Forms out of the Extended Space *

```
           T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1|12|13| 8|  |12|13| 8| 1|  |16| 5| 4| 9|  | 5| 4| 9|16|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |16| 5| 4| 9|  | 5| 4| 9|16|  | 3|10|15| 6|  |10|15| 6| 3|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 3|10|15| 6|  |10|15| 6| 3|  |14| 7| 2|11|  | 7| 2|11|14|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |14| 7| 2|11|  | 7| 2|11|14|  | 1|12|13| 8|  |12|13| 8| 1|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

```
* Make them Smart for n1=1 *
           T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1|12|13| 8|  | 1| 8|13|12|  | 1|12|13| 8|  | 1| 8|13|12|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |16| 5| 4| 9|  |16| 9| 4| 5|  |14| 7| 2|11|  |14|11| 2| 7|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 3|10|15| 6|  | 3| 6|15|10|  | 3|10|15| 6|  | 3| 6|15|10|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |14| 7| 2|11|  |14|11| 2| 7|  |16| 5| 4| 9|  |16| 9| 4| 5|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

```
* Original #3 in Extended Form *    * Moving Frame *
  4  5 16  9  4  5 16  9  4
        .--.--.--.--.                  .--.--.--.--.--.
 15 10| 3| 6|15|10| 3  6 15   13 12| 1| 8|13|12| 1| 8 13
      |--+--+--+--|                |--+--+--+--|--.
  1  8|13|12| 1| 8|13 12  1    2  7|14|11| 2| 7|14|11  2
      |--+--+--+--|                |--+--+--+--|--|
 14 11| 2| 7|14|11| 2  7 14   16  9| 4| 5|16| 9| 4| 5 16
      |--+--+--+--|                |--+--+--+--|--|
  4  5|16| 9| 4| 5|16  9  4    3  6|15|10| 3| 6|15|10  3
      '--'--'--'--'                '--'--'--'--'--'
 15 10  3  6 15 10  3  6 15   13 12| 1| 8|13|12| 1| 8 13
                                   '--'--'--'--'--'
  1  8 13 12  1  8 13 12  1
```

```
* Read 4 Forms out of the Extended Space *
           T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1| 8|13|12|  | 8|13|12| 1|  |14|11| 2| 7|  |11| 2| 7|14|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |14|11| 2| 7|  |11| 2| 7|14|  | 4| 5|16| 9|  | 5|16| 9| 4|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 4| 5|16| 9|  | 5|16| 9| 4|  |15|10| 3| 6|  |10| 3| 6|15|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |15|10| 3| 6|  |10| 3| 6|15|  | 1| 8|13|12|  | 8|13|12| 1|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

```
* Make them Smart for n1=1 *
           T0/            T1/            T2/            T3/
 .--.--.--.--.  .--.--.--.--.  .--.--.--.--.  .--.--.--.--.
 | 1| 8|13|12|  | 1|12|13| 8|  | 1| 8|13|12|  | 1|12|13| 8|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |14|11| 2| 7|  |14| 7| 2|11|  |15|10| 3| 6|  |15| 6| 3|10|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 | 4| 5|16| 9|  | 4| 9|16| 5|  | 4| 5|16| 9|  | 4| 9|16| 5|
 |--+--+--+--|  |--+--+--+--|  |--+--+--+--|  |--+--+--+--|
 |15|10| 3| 6|  |15| 6| 3|10|  |14|11| 2| 7|  |14| 7| 2|11|
 '--'--'--'--'  '--'--'--'--'  '--'--'--'--'  '--'--'--'--'
```

The last example does not only tell how any solution can be transformed, but also tells us how it can be identified with the most fundamental solution with n1=1 of the same group, as you see.

Make any fundamental solution have the inequality conditions as follows:

```
n1=1; n2<n4; n5<n13; and n2<n5
```

How many solutions in all can we find for the most fundamental, then?

The next three lists show the standard solutions and the most fundamental ones of 'Composite' type of semi-magic squares of order 4.

```
** 'Composite' Semi-Magic Squares 4*4:   **
**  List of the 240 Standard Solutions   **
```

			1/				2/				3/				4/				5/				6/
1	8	9	16	1	8	9	16	1	8	10	15	1	8	10	15	1	8	11	14	1	8	11	14
12	13	4	5	12	13	4	5	12	13	3	6	12	13	3	6	12	13	2	7	12	13	2	7
6	3	14	11	7	2	15	10	5	4	14	11	7	2	16	9	5	4	15	10	6	3	16	9
15	10	7	2	14	11	6	3	16	9	7	2	14	11	5	4	16	9	6	3	15	10	5	4

			7/				8/				9/				10/				11/				12/
1	8	9	16	1	8	9	16	1	8	10	15	1	8	10	15	1	8	13	12	1	8	13	12
14	11	6	3	14	11	6	3	14	11	5	4	14	11	5	4	14	11	2	7	14	11	2	7
4	5	12	13	7	2	15	10	3	6	12	13	7	2	16	9	3	6	15	10	4	5	16	9
15	10	7	2	12	13	4	5	16	9	7	2	12	13	3	6	16	9	4	5	15	10	3	6

			13/				14/				15/				16/				17/				18/
1	8	9	16	1	8	9	16	1	8	11	14	1	8	11	14	1	8	13	12	1	8	13	12
15	10	7	2	15	10	7	2	15	10	5	4	15	10	5	4	15	10	3	6	15	10	3	6
4	5	12	13	6	3	14	11	2	7	12	13	6	3	16	9	2	7	14	11	4	5	16	9
14	11	6	3	12	13	4	5	16	9	6	3	12	13	2	7	16	9	4	5	14	11	2	7

			19/				20/				21/				22/				23/				24/
1	8	10	15	1	8	10	15	1	8	11	14	1	8	11	14	1	8	13	12	1	8	13	12
16	9	7	2	16	9	7	2	16	9	6	3	16	9	6	3	16	9	4	5	16	9	4	5
3	6	12	13	5	4	14	11	2	7	12	13	5	4	15	10	2	7	14	11	3	6	15	10
14	11	5	4	12	13	3	6	15	10	5	4	12	13	2	7	15	10	3	6	14	11	2	7

			25/				26/				27/				28/				29/				30/
1	12	5	16	1	12	5	16	1	12	6	15	1	12	6	15	1	12	13	8	1	12	13	8
14	7	10	3	14	7	10	3	14	7	9	4	14	7	9	4	14	7	2	11	14	7	2	11
4	9	8	13	11	2	15	6	3	10	8	13	11	2	16	5	3	10	15	6	4	9	16	5
15	6	11	2	8	13	4	9	16	5	11	2	8	13	3	10	16	5	4	9	15	6	3	10

			31/				32/				33/				34/				35/				36/
1	12	5	16	1	12	5	16	1	12	7	14	1	12	7	14	1	12	13	8	1	12	13	8
15	6	11	2	15	6	11	2	15	6	9	4	15	6	9	4	15	6	3	10	15	6	3	10
4	9	8	13	10	3	14	7	2	11	8	13	10	3	16	5	2	11	14	7	4	9	16	5
14	7	10	3	8	13	4	9	16	5	10	3	8	13	2	11	16	5	4	9	14	7	2	11

			37/				38/				39/				40/				41/				42/
1	12	6	15	1	12	6	15	1	12	7	14	1	12	7	14	1	12	13	8	1	12	13	8
16	5	11	2	16	5	11	2	16	5	10	3	16	5	10	3	16	5	4	9	16	5	4	9
3	10	8	13	9	4	14	7	2	11	8	13	9	4	15	6	2	11	14	7	3	10	15	6
14	7	9	4	8	13	3	10	15	6	9	4	8	13	2	11	15	6	3	10	14	7	2	11

			43/				44/				45/				46/				47/				48/
1	14	3	16	1	14	3	16	1	14	7	12	1	14	7	12	1	14	11	8	1	14	11	8
15	4	13	2	15	4	13	2	15	4	9	6	15	4	9	6	15	4	5	10	15	4	5	10
6	9	8	11	10	5	12	7	2	13	8	11	10	5	16	3	2	13	12	7	6	9	16	3
12	7	10	5	8	11	6	9	16	3	10	5	8	11	2	13	16	3	6	9	12	7	2	13

			49/				50/				51/				52/				53/				54/
1	14	4	15	1	14	4	15	1	14	7	12	1	14	7	12	1	14	11	8	1	14	11	8
16	3	13	2	16	3	13	2	16	3	10	5	16	3	10	5	16	3	6	9	16	3	6	9
5	10	8	11	9	6	12	7	2	13	8	11	9	6	15	4	2	13	12	7	5	10	15	4
12	7	9	6	8	11	5	10	15	4	9	6	8	11	2	13	15	4	5	10	12	7	2	13

			55/				56/				57/				58/				59/				60/
1	15	4	14	1	15	4	14	1	15	6	12	1	15	6	12	1	15	10	8	1	15	10	8
16	2	13	3	16	2	13	3	16	2	11	5	16	2	11	5	16	2	7	9	16	2	7	9
5	11	8	10	9	7	12	6	3	13	8	10	9	7	14	4	3	13	12	6	5	11	14	4
12	6	9	7	8	10	5	11	14	4	9	7	8	10	3	13	14	4	5	11	12	6	3	13

			61/				62/				63/				65/				66/				67/
2	7	9	16	2	7	10	15	2	7	12	13	2	7	9	16	2	7	10	15	2	7	14	11
11	14	4	5	11	14	3	6	11	14	1	8	13	12	6	3	13	12	5	4	13	12	1	8
8	1	15	10	8	1	16	9	5	4	15	10	8	1	15	10	8	1	16	9	3	6	15	10
13	12	6	3	13	12	5	4	16	9	6	3	11	14	4	5	11	14	3	6	16	9	4	5

```
          69/          75/          81/          85/          91/          97/
  2  7  9 16   2  7 10 15   2 11  5 16   2 11  5 16   2 11  6 15   2 13  3 16
 15 10  8  1  16  9  8  1  13  8 10  3  15  6 12  1  16  5 12  1  15  4 14  1
  4  5 11 14   3  6 11 14  12  1 15  6   4  9  7 14   3 10  7 14   6  9  7 12
 13 12  6  3  13 12  5  4   7 14  4  9  13  8 10  3  13  8  9  4  11  8 10  5
         103/         109/         115/         117/         121/         125/
  2 13  4 15   2 15  4 13   3  6 11 14   3  6  9 16   3  6  9 16   3  6 11 14
 16  3 14  1  16  1 14  3  10 15  2  7  13 12  7  2  14 11  8  1  16  9  8  1
  5 10  7 12   5 12  7 10   8  1 16  9   8  1 14 11   7  2 13 12   2  7 10 15
 11  8  9  6  11  6  9  8  13 12  5  4  10 15  4  5  10 15  4  5  13 12  5  4
         131/         135/         139/         145/         151/         157/
  3 10  5 16   3 10  5 16   3 10  7 14   3 13  2 16   3 13  4 14   3 14  4 13
 13  8 11  2  14  7 12  1  16  5 12  1  14  4 15  1  16  2 15  1  16  1 15  2
 12  1 14  7  11  2 13  8   2 11  6 15   7  9  6 12   5 11  6 12   5 12  6 11
  6 15  4  9   6 15  4  9  13  8  9  4  10  8 11  5  10  8  9  7  10  7  9  8
         163/         167/         171/         175/         179/         183/
  4  5 10 15   4  5 10 15   4  5 11 14   4  9  6 15   4  9  6 15   4  9  7 14
 13 12  7  2  14 11  8  1  15 10  8  1  13  8 11  2  14  7 12  1  15  6 12  1
  8  1 14 11   7  2 13 12   6  3 13 12  12  1 14  7  11  2 13  8  10  3 13  8
  9 16  3  6   9 16  3  6   9 16  2  7   5 16  3 10   5 16  3 10   5 16  2 11
         187/         193/         199/         205/         207/         209/
  4 13  2 15   4 13  3 14   4 14  3 13   5  4 13 12   5  4 13 12   5  4 14 11
 14  3 16  1  15  2 16  1  15  1 16  2  10 15  2  7  11 14  3  6  12 13  3  6
  7 10  5 12   6 11  5 12   6 12  5 11   8  1 16  9   8  1 16  9   7  2 16  9
  9  8 11  6   9  8 10  7   9  7 10  8  11 14  3  6  10 15  2  7  10 15  1  8
         211/         217/         219/         221/         223/         225/
  5  4 13 12   5 10  7 12   5 11  6 12   5 12  6 11   6  3 13 12   6  3 14 11
 16  9  8  1  16  3 14  1  16  2 15  1  16  1 15  2  11 14  4  5  12 13  4  5
  2  7 10 15   2 13  4 15   3 13  4 14   3 14  4 13   8  1 15 10   7  2 15 10
 11 14  3  6  11  8  9  6  10  8  9  7  10  7  9  8   9 16  2  7   9 16  1  8
         227/         231/         233/         235/         237/         239/
  6  3 13 12   6 11  5 12   6 12  5 11   7  2 15 10   7  2 15 10   7 12  5 10
 15 10  8  1  15  2 16  1  15  1 16  2  12 13  4  5  14 11  6  3  14  1 16  3
  4  5 11 14   4 13  3 14   4 14  3 13   6  3 14 11   4  5 12 13   4 15  2 13
  9 16  2  7   9  8 10  7   9  7 10  8   9 16  1  8   9 16  1  8   9  6 11  8
```

[Count = 240]

* Counts accordig to the Sum of Diagonals: *

```
(14,54):[  0,  0];   (15,53):[  0,  0];   (16,52):[  0,  0];
(17,51):[  0,  0];   (18,50):[ 48, 48];   (19,49):[  0,  0];
(20,48):[  0,  0];   (21,47):[  0,  0];   (22,46):[  0,  0];
(23,45):[  0,  0];   (24,44):[  0,  0];   (25,43):[  0,  0];
(26,42):[ 36, 36];   (27,41):[  0,  0];   (28,40):[  0,  0];
(29,39):[  0,  0];   (30,38):[ 24, 24];   (31,37):[  0,  0];
(32,36):[ 24, 24];   (33,35):[  0,  0];   (34,34):[ 48, 48];
(35,33):[  0,  0];   (36,32):[ 24, 24];   (37,31):[  0,  0];
(38,30):[ 24, 24];   (39,29):[  0,  0];   (40,28):[  0,  0];
(41,27):[  0,  0];   (42,26):[ 12, 12];   (43,25):[  0,  0];
(44,24):[  0,  0];   (45,23):[  0,  0];   (46,22):[  0,  0];
(47,21):[  0,  0];   (48,20):[  0,  0];   (49,19):[  0,  0];
(50,18):[  0,  0];   (51,17):[  0,  0];   (52,16):[  0,  0];
(53,15):[  0,  0];   (54,14):[  0,  0];   (55,13):[  0,  0];
```

** 'Composite' Semi-Magic Squares 4*4 **
** List of Fundamental 15 Solutions: **

```
          1/          2/          3/          4/          5/
  1  8  9 16   1  8  9 16   1  8 10 15   1  8 10 15   1  8 11 14
 12 13  4  5  12 13  4  5  12 13  3  6  12 13  3  6  12 13  2  7
  6  3 14 11   7  2 15 10   5  4 14 11   7  2 16  9   5  4 15 10
 15 10  7  2  14 11  6  3  16  9  7  2  14 11  5  4  16  9  6  3
          6/          7/          8/          9/         10/
  1  8 11 14   1  8  9 16   1  8 10 15   1  8 13 12   1  8 13 12
 12 13  2  7  14 11  6  3  14 11  5  4  14 11  2  7  14 11  2  7
  6  3 16  9   4  5 12 13   3  6 12 13   3  6 15 10   4  5 16  9
 15 10  5  4  15 10  7  2  16  9  7  2  16  9  4  5  15 10  3  6
         11/         12/         13/         14/         15/
  1  8 11 14   1  8 13 12   1 12  5 16   1 12  6 15   1 12  7 14
 15 10  5  4  15 10  3  6  14  7 10  3  14  7  9  4  15  6  9  4
  2  7 12 13   2  7 14 11   4  9  8 13   3 10  8 13   2 11  8 13
 16  9  6  3  16  9  4  5  15  6 11  2  16  5 11  2  16  5 10  3
```

[Count = 15]

** 'Composite & Complete' Magic Squares 4x4 **
** List of the Fundamental 3 Solutions: **

```
          1/          2/          3/
  1  8 10 15   1  8 11 14   1  8 13 12
 12 13  3  6  12 13  2  7  14 11  2  7
  7  2 16  9   6  3 16  9   4  5 16  9
 14 11  5  4  15 10  5  4  15 10  3  6
```

[Count = 3]

By this method of Extended Space and Moving Frame, you cannot only make each fundamental solution into 4 standard solutions, but can also find which fundamental one any standard one should be identified with, by transforming the standard one back into the fundamental.

On top of that you can kill all the inequality conditions {n1=1; n2<n4; n5<n13; and n2<n5;} and make any fundamental solution into the 'primitive' ones with n1=2, 3, 4, 5, ... , 15, 16. You can also make another reflected form for each.

Therefore you can finally make each fundamental into 4x16x2=128 primitive forms in all. It tells us the reason why our 'Composite' semi-panmagic squares of order 4 have the 1920 primitive solutions and the 15 fundamental ones.

1920=15x128

It is the same reason for 348=3x128 why our 'Composite & Complete' magic squares of order 4 has the solution sets of 348 primitive ones and 3 fundamental.

At the end of this article I feel I must express my deepest appreciation to Prof. Mutsumi Suzuki. He gave me a lot of suggestion and inspiration in 2000-01.

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