Higher Chow groups and Milnor K-theories of DVR

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July 1, 2008

Let R be a discrete valuation ring, k the residue field of R and F the quotient field of R. Bloch [1] and Levine [4] defined the cycle complex $\mathcal{Z}^r(X, \bullet)$ over a B-scheme X of finite type, where B is a regular noetherian scheme of dimension at most one, and proved the existence of the following Localization sequence.

Theorem 1 (Levine [4]) *Let $p : X \to B$ be a scheme of finite type over a regular noetherian scheme B of dimension at most one, and let $i : Z \to X$ be a closed subscheme of X in codimension d, $j : U \to X$ the complement of Z. Then the exact sequence of sheaves on B*

$$0 \to (p{\cdot}i)_*\mathcal{Z}^{r-d}(Z, \bullet) \to p_*\mathcal{Z}^r(X, \bullet) \to (p{\cdot}j)_*\mathcal{Z}^r(U, \bullet)$$

is a distinguished triangle. Furthermore, if B is a semilocal PID, then $\mathcal{Z}^r(U, *)/j^*\mathcal{Z}^r(X, *)$ is acyclic, hence the sequence of the cycle complexes

$$0 \to \mathcal{Z}^{r-d}(Z, \bullet) \xrightarrow{i_*} \mathcal{Z}^r(X, \bullet) \xrightarrow{j^*} \mathcal{Z}^r(U, \bullet)$$

is a distinguished triangle.

On the other hand, Totaro [6] proved that the zero dimensional higher Chow group $\mathrm{CH}^n(F, n)$ and the Minor K-group $K_n^M(F)$ are isomorphic for arbitrary field F, and the isomorphism ψ is constructed by the norm map.

The purpose of this talk is to construct $\psi : \mathrm{CH}^n(R, n) \to K_n^M(R)$, and we prove the following Main Theorem.

Theorem 2 *Let R be a 6-fold stable discrete valuation ring, k the residue field of R and F the quotient field of R. Then we obtain a canonical map $\psi : \mathrm{CH}^n(R, n) \to K_n^M(R)$. Furthermore, ψ is surjective, and the kernel of ψ coincides with the image of $i_* : \mathrm{CH}^{n-1}(k, n) \to \mathrm{CH}^n(R, n)$.*

As its application, we can prove Gersten's conjecture for Milnor K-theory of 6-fold stable discrete valuation rings.

Conjecture 3 (Gersten's conjecture for Milnor K-theories of DVR) *Let R be an arbitrary discrete valuation ring, k the residue field, F be the quotient field. Then there exists the following short exact sequence:*

$$0 \to K_n^M(R) \xrightarrow{j_M^*} K_n^M(F) \xrightarrow{\partial_\pi} K_{n-1}^M(k) \to 0.$$

This Conjecture is implied by the Beilinson conjectures. In this case the only problem is to check that the map $j_M^* : K_n^M(R) \to K_n^M(F)$ is injective, and this conjecture proved by Suslin and Yarosh [5] in case the discrete valuation ring is geometric type over an infinite field and $n = 3$. Here, we obtain the following as a solution of the conjecture on condition that R is 6-fold stable.

Corollary 4 *Let R be a 6-fold stable discrete valuation ring and F the quotient field of R. Then we obtain the map $j_M^* : K_n^M(R) \to K_n^M(F)$ is injective.*

References

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